xAct‘xPert‘

This is the doc file xPertDoc.nb of version 1.0.0 of `xPert‘`. Last update on 30 June 2008.

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Intro

`xPert‘` is a package for efficient construction and manipulation of high−order perturbations of the curvature tensors of a Riemannian manifold. Its main application is Perturbation Theory in General Relativity.

`xPert‘` is part of the `xAct‘` framework for Tensor Computer Algebra in *Mathematica*, and in particular it needs the underlying package `xTensor‘` for abstract tensor manipulations.

Load the package

This loads the package from the default directory, for example $Home/.*Mathematica*/Applications/xAct/ for a single−user installa−tion under Linux. `xTensor‘` and `xPerm‘` are automatically loaded.

```
In[1]:= MemoryInUse[]

Out[1]= 2125968
```

```
In[2]:= <<xAct`xPert`

--------------------------------------------------------------------------------

Package xAct`xCore`  version 0.6.0, {2008, 6, 30}

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Package ExpressionManipulation`

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Package xAct`xPerm`  version 1.0.1, {2008, 5, 16}

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Connecting to external linux executable...

Connection established.

--------------------------------------------------------------------------------

Package xAct`xTensor`  version 0.9.6, {2008, 6, 30}

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--------------------------------------------------------------------------------
```

Comparing, we see that the packages take about 10Mb in *Mathematica* 5.2:

```
In[3]:= MemoryInUse[]

Out[3]= 11734032

In[4]:= Out[3] - Out[1]

Out[4]= 9608064
```

There are seven contexts: `xAct‘xPert‘`, `xAct‘xTensor‘`, `xAct‘xPerm‘`, `xAct‘xCore‘` and `xAct‘ExpressionManipulation‘` contain the respective reserved words. `System‘` contains *Mathematica*'s reserved words. The current context `Global‘` will contain your definitions and right now it is empty.

```
In[5]:= $ContextPath

Out[5]= {xAct‘xPert‘, xAct‘xTensor‘, xAct‘xPerm‘,
         xAct‘xCore‘, xAct‘ExpressionManipulation‘, Global‘, System‘}

In[6]:= Context[]

Out[6]= Global‘

In[7]:= ?Global‘*

        Information::nomatch : No symbol matching Global‘* found. More…
```

The default options of `MakeRule` and `ContractMetric` have been changed:

```
In[8]:= Options[MakeRule]

Out[8]= {PatternIndices → All, TestIndices → True, MetricOn → All,
         UseSymmetries → True, Verbose → False, ContractMetrics → True}

In[9]:= Options[ContractMetric]

Out[9]= {AllowUpperDerivatives → True, OverDerivatives → False}
```

We also load the timing package and lower its default threshold of 1 second to 0.1 seconds

```
In[10]:= << xAct‘ShowTime1‘

In[11]:= $ShowTimeThreshold = 0.1;
```

We define a function to collect and canonicalize equal−order terms:

```
In[12]:= org[expr_] := Collect[ContractMetric[expr], $PerturbationParameter, ToCanonical]
```

0. Set up

`xPert‘` has been built on `xTensor‘` and shares its notation. Then, a session with `xPert‘` begins just like one with `xTensor‘`, defining one or more manifolds and vector bundles and the objects living on them:

Define a 4d manifold `M`. Note that most of the results will not depend on the chosen dimension.

```
In[13]:= DefManifold[M, 4, {a, b, c, d, e, f}]

           ** DefManifold: Defining manifold M.

           ** DefVBundle: Defining vbundle TangentM.
```

We also define a metric on this manifold, with negative determinant and associated covariant derivative CD. This derivative is extended to act on tensor densities defined using the determinant of the metric g in the basis AIndex:

```
In[14]:= DefMetric[-1, g[-a, -b], CD, {";", "D"}, WeightedWithBasis → AIndex]

** DefTensor: Defining symmetric metric tensor g[-a, -b].
** DefTensor: Defining antisymmetric tensor epsilong[a, b, c, d].
** DefCovD: Defining covariant derivative CD[-a].
** DefTensor: Defining vanishing torsion tensor TorsionCD[a, -b, -c].
** DefTensor: Defining symmetric Christoffel tensor ChristoffelCD[a, -b, -c].
** DefTensor: Defining Riemann tensor RiemannCD[-a, -b, -c, -d].
** DefTensor: Defining symmetric Ricci tensor RicciCD[-a, -b].
** DefCovD:  Contractions of Riemann automatically replaced by Ricci.
** DefTensor: Defining Ricci scalar RicciScalarCD[].
** DefCovD:  Contractions of Ricci automatically replaced by RicciScalar.
** DefTensor: Defining symmetric Einstein tensor EinsteinCD[-a, -b].
** DefTensor: Defining Weyl tensor WeylCD[-a, -b, -c, -d].
   Rules {1, 2, 3, 4, 5, 6, 7, 8} have been declared as DownValues for WeylCD.
** DefTensor: Defining symmetric TFRicci tensor TFRicciCD[-a, -b].
   Rules {1, 2} have been declared as DownValues for TFRicciCD.
** DefCovD:  Computing RiemannToWeylRules for dim 4
** DefCovD:  Computing RicciToTFRicci for dim 4
** DefCovD:  Computing RicciToEinsteinRules for dim 4
** DefTensor: Defining weight +2 density Detg[]. Determinant.
** DefTensor: Defining tensor Tetrag[-a, -b, -c, -d].
** DefTensor: Defining tensor Tetrag†[-a, -b, -c, -d].
0.788049 Second
```

The absolute value of its determinant in the basis AIndex is obtained as

```
In[15]:= Determinant[g][]

Out[15]= g̃
```

and it is actually called

```
In[16]:= InputForm[%]

Out[16]//InputForm=
     Detg[]
```

By default, curvature tensors are output showing the name of the derivative they are associated to:

```
In[17]:= RiemannCD[-a, -b, -c, d]

Out[17]= R[D]_abc^d
```

Because we shall work with a single covariant derivative operator, we can choose a simpler output for the geometric tensors

```
In[18]:= PrintAs[RiemannCD] ^= "R";
         PrintAs[RicciCD] ^= "R";
         PrintAs[RicciScalarCD] ^= "R";
         PrintAs[ChristoffelCD] ^= "Γ";
         PrintAs[EinsteinCD] ^= "G";
```

so that now

```
In[23]:= RiemannCD[-a, -b, -c, d]

Out[23]= R_abc^d
```

After defining the background manifold in `xTensor‘`, we now define the perturbative structure with the command `DefMetricPerturbation` of `xPert‘`.

We define the symmetric two−tensor `h` that will be understood as the perturbation of the metric. As can be seen in the output, the first index of this tensor is a label index (`LI`), which will encode the perturbative order.

```
In[24]:= DefMetricPerturbation[g, h, ϵ]

      ** DefParameter: Defining parameter ϵ.

      ** DefTensor: Defining tensor h[LI[order], -a, -b].
```

The perturbative parameter ϵ is stored in the global variable

```
In[25]:= $PerturbationParameter

Out[25]= ϵ
```

Finally, we colour in blue the perturbative orders

```
In[26]:= Unprotect[IndexForm];
         IndexForm[LI[x_]] := ColorString[ToString[x], RGBColor[0, 0, 1]];
         Protect[IndexForm];

In[29]:= IndexForm[LI[n]]

Out[29]= n
```

■ 1. Perturbations of the metric and fundamental properties

The main objective of `xPert‘` is giving the *n*−th perturbation of any object in terms of metric perturbations of order *n* and lower. Here we introduce the `Perturbation` head and give examples of use of different perturbative orders. For those readers already familiar with `xTensor‘`, let us point out that the symbol `Perturbation` is a linear inert−head.

The perturbation of the metric is represented as follows:

```
In[30]:= Perturbation[ g[-a, -b] ]
```

Out[30]= $h^{1}{}_{ab}$

`Perturbation` has actually two argumens in general. The first one is the expression we want to perturb and the second one the perturbative order (with default 1 when it is not specified, as in the previous case). For instance, acting on the metric at fifth perturbative order returns

```
In[31]:= Perturbation[g[-a, -b], 5]
```

Out[31]= $h^{5}{}_{ab}$

Or on the perturbed tensor itself,

```
In[32]:= Perturbation[h[LI[3], -a, -b], 5]
```

Out[32]= $h^{8}{}_{ab}$

It also knows how to act on a Kronecker delta

```
In[33]:= Perturbation[delta[a, -b], 3]
```

Out[33]= 0

`Perturbation` uses the fundamental properties of a derivative: linearity and Leibnitz rule. When it does not know the answer of the action on a given tensor, for example the inverse of the metric or the Ricci tensor, the head `Perturbation` is kept (this is where its inert−head character plays a role):

```
In[34]:= Perturbation[g[-a, -b] + g[c, d] RicciCD[-a, -c] RicciCD[-d, -b]]
```

Out[34]= $h^{1}{}_{ab} + g^{cd} \triangle[R_{db}] R_{ac} + g^{cd} \triangle[R_{ac}] R_{db} + \triangle[g^{cd}] R_{ac} R_{db}$

We can consider the same case but at a higher perturbative order, for instance at seveteenth order and see that the calculation is almost immediate.

```
In[35]:= Perturbation[g[-a, -b] + g[c, d] RicciCD[-a, -c] RicciCD[-d, -b], 17]
```

Out[35]=
$$h^{17}{}_{ab} + 272 \triangle[R_{ac}] \triangle[R_{db}] \triangle^{15}[g^{cd}] + 2040 \triangle[R_{db}] \triangle^{14}[g^{cd}] \triangle^{2}[R_{ac}] +$$
$$9520 \triangle[R_{db}] \triangle^{13}[g^{cd}] \triangle^{3}[R_{ac}] + 30940 \triangle[R_{db}] \triangle^{12}[g^{cd}] \triangle^{4}[R_{ac}] +$$
$$74256 \triangle[R_{db}] \triangle^{11}[g^{cd}] \triangle^{5}[R_{ac}] + 136136 \triangle[R_{db}] \triangle^{10}[g^{cd}] \triangle^{6}[R_{ac}] +$$
$$194480 \triangle[R_{db}] \triangle^{9}[g^{cd}] \triangle^{7}[R_{ac}] + 218790 \triangle[R_{db}] \triangle^{8}[g^{cd}] \triangle^{8}[R_{ac}] +$$
$$194480 \triangle[R_{db}] \triangle^{7}[g^{cd}] \triangle^{9}[R_{ac}] + 136136 \triangle[R_{db}] \triangle^{6}[g^{cd}] \triangle^{10}[R_{ac}] +$$
$$74256 \triangle[R_{db}] \triangle^{5}[g^{cd}] \triangle^{11}[R_{ac}] + 30940 \triangle[R_{db}] \triangle^{4}[g^{cd}] \triangle^{12}[R_{ac}] +$$
$$9520 \triangle[R_{db}] \triangle^{3}[g^{cd}] \triangle^{13}[R_{ac}] + 2040 \triangle[R_{db}] \triangle^{2}[g^{cd}] \triangle^{14}[R_{ac}] + 272 \triangle[g^{cd}] \triangle[R_{db}] \triangle^{15}[R_{ac}] +$$
$$17\, g^{cd} \triangle[R_{db}] \triangle^{16}[R_{ac}] + 2040 \triangle[R_{ac}] \triangle^{14}[g^{cd}] \triangle^{2}[R_{db}] + 14280 \triangle^{13}[g^{cd}] \triangle^{2}[R_{ac}] \triangle^{2}[R_{db}] +$$
$$61880 \triangle^{12}[g^{cd}] \triangle^{3}[R_{ac}] \triangle^{2}[R_{db}] + 185640 \triangle^{11}[g^{cd}] \triangle^{4}[R_{ac}] \triangle^{2}[R_{db}] +$$
$$408408 \triangle^{10}[g^{cd}] \triangle^{5}[R_{ac}] \triangle^{2}[R_{db}] + 680680 \triangle^{9}[g^{cd}] \triangle^{6}[R_{ac}] \triangle^{2}[R_{db}] +$$
$$875160 \triangle^{8}[g^{cd}] \triangle^{7}[R_{ac}] \triangle^{2}[R_{db}] + 875160 \triangle^{7}[g^{cd}] \triangle^{8}[R_{ac}] \triangle^{2}[R_{db}] +$$
$$680680 \triangle^{6}[g^{cd}] \triangle^{9}[R_{ac}] \triangle^{2}[R_{db}] + 408408 \triangle^{5}[g^{cd}] \triangle^{10}[R_{ac}] \triangle^{2}[R_{db}] +$$
$$185640 \triangle^{4}[g^{cd}] \triangle^{11}[R_{ac}] \triangle^{2}[R_{db}] + 61880 \triangle^{3}[g^{cd}] \triangle^{12}[R_{ac}] \triangle^{2}[R_{db}] +$$
$$14280 \triangle^{2}[g^{cd}] \triangle^{13}[R_{ac}] \triangle^{2}[R_{db}] + 2040 \triangle[g^{cd}] \triangle^{14}[R_{ac}] \triangle^{2}[R_{db}] +$$

$$136\, g^{cd}\, \triangle^{15}[R_{ac}]\, \triangle^{2}[R_{db}] + 9520\, \triangle[R_{ac}]\, \triangle^{13}[g^{cd}]\, \triangle^{3}[R_{db}] + 61880\, \triangle^{12}[g^{cd}]\, \triangle^{2}[R_{ac}]\, \triangle^{3}[R_{db}] +$$
$$247520\, \triangle^{11}[g^{cd}]\, \triangle^{3}[R_{ac}]\, \triangle^{3}[R_{db}] + 680680\, \triangle^{10}[g^{cd}]\, \triangle^{4}[R_{ac}]\, \triangle^{3}[R_{db}] +$$
$$1361360\, \triangle^{9}[g^{cd}]\, \triangle^{5}[R_{ac}]\, \triangle^{3}[R_{db}] + 2042040\, \triangle^{8}[g^{cd}]\, \triangle^{6}[R_{ac}]\, \triangle^{3}[R_{db}] +$$
$$2333760\, \triangle^{7}[g^{cd}]\, \triangle^{7}[R_{ac}]\, \triangle^{3}[R_{db}] + 2042040\, \triangle^{6}[g^{cd}]\, \triangle^{8}[R_{ac}]\, \triangle^{3}[R_{db}] +$$
$$1361360\, \triangle^{5}[g^{cd}]\, \triangle^{9}[R_{ac}]\, \triangle^{3}[R_{db}] + 680680\, \triangle^{4}[g^{cd}]\, \triangle^{10}[R_{ac}]\, \triangle^{3}[R_{db}] +$$
$$247520\, \triangle^{3}[g^{cd}]\, \triangle^{11}[R_{ac}]\, \triangle^{3}[R_{db}] + 61880\, \triangle^{2}[g^{cd}]\, \triangle^{12}[R_{ac}]\, \triangle^{3}[R_{db}] +$$
$$9520\, \triangle[g^{cd}]\, \triangle^{13}[R_{ac}]\, \triangle^{3}[R_{db}] + 680\, g^{cd}\, \triangle^{14}[R_{ac}]\, \triangle^{3}[R_{db}] + 30940\, \triangle[R_{ac}]\, \triangle^{12}[g^{cd}]\, \triangle^{4}[R_{db}] +$$
$$185640\, \triangle^{11}[g^{cd}]\, \triangle^{2}[R_{ac}]\, \triangle^{4}[R_{db}] + 680680\, \triangle^{10}[g^{cd}]\, \triangle^{3}[R_{ac}]\, \triangle^{4}[R_{db}] +$$
$$1701700\, \triangle^{9}[g^{cd}]\, \triangle^{4}[R_{ac}]\, \triangle^{4}[R_{db}] + 3063060\, \triangle^{8}[g^{cd}]\, \triangle^{5}[R_{ac}]\, \triangle^{4}[R_{db}] +$$
$$4084080\, \triangle^{7}[g^{cd}]\, \triangle^{6}[R_{ac}]\, \triangle^{4}[R_{db}] + 4084080\, \triangle^{6}[g^{cd}]\, \triangle^{7}[R_{ac}]\, \triangle^{4}[R_{db}] +$$
$$3063060\, \triangle^{5}[g^{cd}]\, \triangle^{8}[R_{ac}]\, \triangle^{4}[R_{db}] + 1701700\, \triangle^{4}[g^{cd}]\, \triangle^{9}[R_{ac}]\, \triangle^{4}[R_{db}] +$$
$$680680\, \triangle^{3}[g^{cd}]\, \triangle^{10}[R_{ac}]\, \triangle^{4}[R_{db}] + 185640\, \triangle^{2}[g^{cd}]\, \triangle^{11}[R_{ac}]\, \triangle^{4}[R_{db}] +$$
$$30940\, \triangle[g^{cd}]\, \triangle^{12}[R_{ac}]\, \triangle^{4}[R_{db}] + 2380\, g^{cd}\, \triangle^{13}[R_{ac}]\, \triangle^{4}[R_{db}] + 74256\, \triangle[R_{ac}]\, \triangle^{11}[g^{cd}]\, \triangle^{5}[R_{db}] +$$
$$408408\, \triangle^{10}[g^{cd}]\, \triangle^{2}[R_{ac}]\, \triangle^{5}[R_{db}] + 1361360\, \triangle^{9}[g^{cd}]\, \triangle^{3}[R_{ac}]\, \triangle^{5}[R_{db}] +$$
$$3063060\, \triangle^{8}[g^{cd}]\, \triangle^{4}[R_{ac}]\, \triangle^{5}[R_{db}] + 4900896\, \triangle^{7}[g^{cd}]\, \triangle^{5}[R_{ac}]\, \triangle^{5}[R_{db}] +$$
$$5717712\, \triangle^{6}[g^{cd}]\, \triangle^{6}[R_{ac}]\, \triangle^{5}[R_{db}] + 4900896\, \triangle^{5}[g^{cd}]\, \triangle^{7}[R_{ac}]\, \triangle^{5}[R_{db}] +$$
$$3063060\, \triangle^{4}[g^{cd}]\, \triangle^{8}[R_{ac}]\, \triangle^{5}[R_{db}] + 1361360\, \triangle^{3}[g^{cd}]\, \triangle^{9}[R_{ac}]\, \triangle^{5}[R_{db}] +$$
$$408408\, \triangle^{2}[g^{cd}]\, \triangle^{10}[R_{ac}]\, \triangle^{5}[R_{db}] + 74256\, \triangle[g^{cd}]\, \triangle^{11}[R_{ac}]\, \triangle^{5}[R_{db}] + 6188\, g^{cd}\, \triangle^{12}[R_{ac}]\, \triangle^{5}[R_{db}] +$$
$$136136\, \triangle[R_{ac}]\, \triangle^{10}[g^{cd}]\, \triangle^{6}[R_{db}] + 680680\, \triangle^{9}[g^{cd}]\, \triangle^{2}[R_{ac}]\, \triangle^{6}[R_{db}] +$$
$$2042040\, \triangle^{8}[g^{cd}]\, \triangle^{3}[R_{ac}]\, \triangle^{6}[R_{db}] + 4084080\, \triangle^{7}[g^{cd}]\, \triangle^{4}[R_{ac}]\, \triangle^{6}[R_{db}] +$$
$$5717712\, \triangle^{6}[g^{cd}]\, \triangle^{5}[R_{ac}]\, \triangle^{6}[R_{db}] + 5717712\, \triangle^{5}[g^{cd}]\, \triangle^{6}[R_{ac}]\, \triangle^{6}[R_{db}] +$$
$$4084080\, \triangle^{4}[g^{cd}]\, \triangle^{7}[R_{ac}]\, \triangle^{6}[R_{db}] + 2042040\, \triangle^{3}[g^{cd}]\, \triangle^{8}[R_{ac}]\, \triangle^{6}[R_{db}] +$$
$$680680\, \triangle^{2}[g^{cd}]\, \triangle^{9}[R_{ac}]\, \triangle^{6}[R_{db}] + 136136\, \triangle[g^{cd}]\, \triangle^{10}[R_{ac}]\, \triangle^{6}[R_{db}] +$$
$$12376\, g^{cd}\, \triangle^{11}[R_{ac}]\, \triangle^{6}[R_{db}] + 194480\, \triangle[R_{ac}]\, \triangle^{9}[g^{cd}]\, \triangle^{7}[R_{db}] + 875160\, \triangle^{8}[g^{cd}]\, \triangle^{2}[R_{ac}]\, \triangle^{7}[R_{db}] +$$
$$2333760\, \triangle^{7}[g^{cd}]\, \triangle^{3}[R_{ac}]\, \triangle^{7}[R_{db}] + 4084080\, \triangle^{6}[g^{cd}]\, \triangle^{4}[R_{ac}]\, \triangle^{7}[R_{db}] +$$
$$4900896\, \triangle^{5}[g^{cd}]\, \triangle^{5}[R_{ac}]\, \triangle^{7}[R_{db}] + 4084080\, \triangle^{4}[g^{cd}]\, \triangle^{6}[R_{ac}]\, \triangle^{7}[R_{db}] +$$
$$2333760\, \triangle^{3}[g^{cd}]\, \triangle^{7}[R_{ac}]\, \triangle^{7}[R_{db}] + 875160\, \triangle^{2}[g^{cd}]\, \triangle^{8}[R_{ac}]\, \triangle^{7}[R_{db}] +$$
$$194480\, \triangle[g^{cd}]\, \triangle^{9}[R_{ac}]\, \triangle^{7}[R_{db}] + 19448\, g^{cd}\, \triangle^{10}[R_{ac}]\, \triangle^{7}[R_{db}] + 218790\, \triangle[R_{ac}]\, \triangle^{8}[g^{cd}]\, \triangle^{8}[R_{db}] +$$
$$875160\, \triangle^{7}[g^{cd}]\, \triangle^{2}[R_{ac}]\, \triangle^{8}[R_{db}] + 2042040\, \triangle^{6}[g^{cd}]\, \triangle^{3}[R_{ac}]\, \triangle^{8}[R_{db}] +$$
$$3063060\, \triangle^{5}[g^{cd}]\, \triangle^{4}[R_{ac}]\, \triangle^{8}[R_{db}] + 3063060\, \triangle^{4}[g^{cd}]\, \triangle^{5}[R_{ac}]\, \triangle^{8}[R_{db}] +$$
$$2042040\, \triangle^{3}[g^{cd}]\, \triangle^{6}[R_{ac}]\, \triangle^{8}[R_{db}] + 875160\, \triangle^{2}[g^{cd}]\, \triangle^{7}[R_{ac}]\, \triangle^{8}[R_{db}] +$$
$$218790\, \triangle[g^{cd}]\, \triangle^{8}[R_{ac}]\, \triangle^{8}[R_{db}] + 24310\, g^{cd}\, \triangle^{9}[R_{ac}]\, \triangle^{8}[R_{db}] + 194480\, \triangle[R_{ac}]\, \triangle^{7}[g^{cd}]\, \triangle^{9}[R_{db}] +$$
$$680680\, \triangle^{6}[g^{cd}]\, \triangle^{2}[R_{ac}]\, \triangle^{9}[R_{db}] + 1361360\, \triangle^{5}[g^{cd}]\, \triangle^{3}[R_{ac}]\, \triangle^{9}[R_{db}] +$$
$$1701700\, \triangle^{4}[g^{cd}]\, \triangle^{4}[R_{ac}]\, \triangle^{9}[R_{db}] + 1361360\, \triangle^{3}[g^{cd}]\, \triangle^{5}[R_{ac}]\, \triangle^{9}[R_{db}] +$$
$$680680\, \triangle^{2}[g^{cd}]\, \triangle^{6}[R_{ac}]\, \triangle^{9}[R_{db}] + 194480\, \triangle[g^{cd}]\, \triangle^{7}[R_{ac}]\, \triangle^{9}[R_{db}] + 24310\, g^{cd}\, \triangle^{8}[R_{ac}]\, \triangle^{9}[R_{db}] +$$
$$136136\, \triangle[R_{ac}]\, \triangle^{6}[g^{cd}]\, \triangle^{10}[R_{db}] + 408408\, \triangle^{5}[g^{cd}]\, \triangle^{2}[R_{ac}]\, \triangle^{10}[R_{db}] +$$
$$680680\, \triangle^{4}[g^{cd}]\, \triangle^{3}[R_{ac}]\, \triangle^{10}[R_{db}] + 680680\, \triangle^{3}[g^{cd}]\, \triangle^{4}[R_{ac}]\, \triangle^{10}[R_{db}] +$$
$$408408\, \triangle^{2}[g^{cd}]\, \triangle^{5}[R_{ac}]\, \triangle^{10}[R_{db}] + 136136\, \triangle[g^{cd}]\, \triangle^{6}[R_{ac}]\, \triangle^{10}[R_{db}] +$$
$$19448\, g^{cd}\, \triangle^{7}[R_{ac}]\, \triangle^{10}[R_{db}] + 74256\, \triangle[R_{ac}]\, \triangle^{5}[g^{cd}]\, \triangle^{11}[R_{db}] +$$
$$185640\, \triangle^{4}[g^{cd}]\, \triangle^{2}[R_{ac}]\, \triangle^{11}[R_{db}] + 247520\, \triangle^{3}[g^{cd}]\, \triangle^{3}[R_{ac}]\, \triangle^{11}[R_{db}] +$$
$$185640\, \triangle^{2}[g^{cd}]\, \triangle^{4}[R_{ac}]\, \triangle^{11}[R_{db}] + 74256\, \triangle[g^{cd}]\, \triangle^{5}[R_{ac}]\, \triangle^{11}[R_{db}] +$$
$$12376\, g^{cd}\, \triangle^{6}[R_{ac}]\, \triangle^{11}[R_{db}] + 30940\, \triangle[R_{ac}]\, \triangle^{4}[g^{cd}]\, \triangle^{12}[R_{db}] + 61880\, \triangle^{3}[g^{cd}]\, \triangle^{2}[R_{ac}]\, \triangle^{12}[R_{db}] +$$
$$61880\, \triangle^{2}[g^{cd}]\, \triangle^{3}[R_{ac}]\, \triangle^{12}[R_{db}] + 30940\, \triangle[g^{cd}]\, \triangle^{4}[R_{ac}]\, \triangle^{12}[R_{db}] +$$
$$6188\, g^{cd}\, \triangle^{5}[R_{ac}]\, \triangle^{12}[R_{db}] + 9520\, \triangle[R_{ac}]\, \triangle^{3}[g^{cd}]\, \triangle^{13}[R_{db}] + 14280\, \triangle^{2}[g^{cd}]\, \triangle^{2}[R_{ac}]\, \triangle^{13}[R_{db}] +$$
$$9520\, \triangle[g^{cd}]\, \triangle^{3}[R_{ac}]\, \triangle^{13}[R_{db}] + 2380\, g^{cd}\, \triangle^{4}[R_{ac}]\, \triangle^{13}[R_{db}] + 2040\, \triangle[R_{ac}]\, \triangle^{2}[g^{cd}]\, \triangle^{14}[R_{db}] +$$
$$2040\, \triangle[g^{cd}]\, \triangle^{2}[R_{ac}]\, \triangle^{14}[R_{db}] + 680\, g^{cd}\, \triangle^{3}[R_{ac}]\, \triangle^{14}[R_{db}] + 272\, \triangle[g^{cd}]\, \triangle[R_{ac}]\, \triangle^{15}[R_{db}] +$$
$$136\, g^{cd}\, \triangle^{2}[R_{ac}]\, \triangle^{15}[R_{db}] + 17\, g^{cd}\, \triangle[R_{ac}]\, \triangle^{16}[R_{db}] + 17\, \triangle[R_{db}]\, \triangle^{16}[g^{cd}]\, R_{ac} +$$
$$136\, \triangle^{15}[g^{cd}]\, \triangle^{2}[R_{db}]\, R_{ac} + 680\, \triangle^{14}[g^{cd}]\, \triangle^{3}[R_{db}]\, R_{ac} + 2380\, \triangle^{13}[g^{cd}]\, \triangle^{4}[R_{db}]\, R_{ac} +$$
$$6188\, \triangle^{12}[g^{cd}]\, \triangle^{5}[R_{db}]\, R_{ac} + 12376\, \triangle^{11}[g^{cd}]\, \triangle^{6}[R_{db}]\, R_{ac} + 19448\, \triangle^{10}[g^{cd}]\, \triangle^{7}[R_{db}]\, R_{ac} +$$
$$24310\, \triangle^{9}[g^{cd}]\, \triangle^{8}[R_{db}]\, R_{ac} + 24310\, \triangle^{8}[g^{cd}]\, \triangle^{9}[R_{db}]\, R_{ac} + 19448\, \triangle^{7}[g^{cd}]\, \triangle^{10}[R_{db}]\, R_{ac} +$$
$$12376\, \triangle^{6}[g^{cd}]\, \triangle^{11}[R_{db}]\, R_{ac} + 6188\, \triangle^{5}[g^{cd}]\, \triangle^{12}[R_{db}]\, R_{ac} + 2380\, \triangle^{4}[g^{cd}]\, \triangle^{13}[R_{db}]\, R_{ac} +$$
$$680\, \triangle^{3}[g^{cd}]\, \triangle^{14}[R_{db}]\, R_{ac} + 136\, \triangle^{2}[g^{cd}]\, \triangle^{15}[R_{db}]\, R_{ac} + 17\, \triangle[g^{cd}]\, \triangle^{16}[R_{db}]\, R_{ac} +$$
$$g^{cd}\, \triangle^{17}[R_{db}]\, R_{ac} + 17\, \triangle[R_{ac}]\, \triangle^{16}[g^{cd}]\, R_{db} + 136\, \triangle^{15}[g^{cd}]\, \triangle^{2}[R_{ac}]\, R_{db} +$$

$680\,\triangle^{14}[g^{cd}]\,\triangle^{3}[R_{ac}]\,R_{db} + 2380\,\triangle^{13}[g^{cd}]\,\triangle^{4}[R_{ac}]\,R_{db} + 6188\,\triangle^{12}[g^{cd}]\,\triangle^{5}[R_{ac}]\,R_{db} + 12376\,\triangle^{11}[g^{cd}]\,\triangle^{6}[R_{ac}]\,R_{db} + 19448\,\triangle^{10}[g^{cd}]\,\triangle^{7}[R_{ac}]\,R_{db} + 24310\,\triangle^{9}[g^{cd}]\,\triangle^{8}[R_{ac}]\,R_{db} + 24310\,\triangle^{8}[g^{cd}]\,\triangle^{9}[R_{ac}]\,R_{db} + 19448\,\triangle^{7}[g^{cd}]\,\triangle^{10}[R_{ac}]\,R_{db} + 12376\,\triangle^{6}[g^{cd}]\,\triangle^{11}[R_{ac}]\,R_{db} + 6188\,\triangle^{5}[g^{cd}]\,\triangle^{12}[R_{ac}]\,R_{db} + 2380\,\triangle^{4}[g^{cd}]\,\triangle^{13}[R_{ac}]\,R_{db} + 680\,\triangle^{3}[g^{cd}]\,\triangle^{14}[R_{ac}]\,R_{db} + 136\,\triangle^{2}[g^{cd}]\,\triangle^{15}[R_{ac}]\,R_{db} + 17\,\triangle[g^{cd}]\,\triangle^{16}[R_{ac}]\,R_{db} + g^{cd}\,\triangle^{17}[R_{ac}]\,R_{db} + \triangle^{17}[g^{cd}]\,R_{ac}\,R_{db}$

The perturbative order of an expression can be computed with

```
In[36]:= PerturbationOrder[%]
```

Out[36]= 17

By definition, `Perturbation` commutes with partial derivatives,

```
In[37]:= Perturbation[PD[-a][RicciCD[-c, -d]], 5]
```

Out[37]= $\partial_a\,\triangle^5[R_{cd}]$

and with covariant derivatives acting on scalars,

```
In[38]:= Perturbation[CD[-a][RicciScalarCD[]], 6]
```

Out[38]= $D_a\,\triangle^6[R]$

but not with covariant derivatives acting on tensors,

```
In[39]:= Perturbation[CD[-a][RicciCD[a, -b]], 3]
```

Out[39]= $\triangle^3[R^{a}{}_{b;a}]$

It is essential to have in mind that the `ToCanonical` or `Simplification` commands cannot be used as long as there is any `Perturbation` head in the expression because internal indices of the expression inside `Perturbation` could be moved improperly. A warning reminds us of the danger:

For example the expression

```
In[40]:= h[LI[3], -a, -e] CD[-d][Perturbation[RicciCD[a, e]]]
```

Out[40]= $h^{3}{}_{ae}\,(D_d\,\triangle[R^{ae}])$

is not properly canonicalized here:

```
In[41]:= % // ToCanonical

ToCanonical::canonpert :
 ToCanonical on Perturbation may incorrectly change index characters.
```

Out[41]= $h^{3\,ae}\,(D_d\,\triangle[R_{ae}])$

This is not a restriction at all. It just means that the perturbations must be replaced by other tensor expression before canonicalization. How this is done is shown next. In case canonicalization at this stage is really needed the right sequence of commands is

```
In[42]:= ToCanonical[%% // SeparateMetric[] // ContractMetric, UseMetricOnVBundle → None]
```

Out[42]= $2\, h^3{}_{ae}\, R^a{}_b\, (D_d\, \triangle[g^{eb}]) + h^{3\,ae}\, (D_d\, \triangle[R_{ae}]) + 2\, h^3{}_{ae}\, \triangle[g^{ab}]\, R^e{}_{b;d}$

which uses explicit metric factors instead of moving indices up and down.

2. ExpandPerturbation

The command `ExpandPerturbation` is the most powerful one of this package. As we have seen, the `Perturbation` operator automatically uses linearity and the Leibnitz rule to expand expressions. But up to now it only knows how to act on the metric with covariant indices. All other cases are handled by `ExpandPerturbation`, which makes use of exact formulas to expand the perturbations of any geometric object.

The perturbations of the inverse metric are

```
In[43]:= Perturbation[g[a, b]]
```

Out[43]= $\triangle[g^{ab}]$

```
In[44]:= % // ExpandPerturbation
```

Out[44]= $-h^{1\,ab}$

```
In[45]:= Perturbation[g[a, b], 4]
```

Out[45]= $\triangle^4[g^{ab}]$

```
In[46]:= % // ExpandPerturbation
```

Out[46]= $24\, h^{1\,ac}\, h^1{}_c{}^d\, h^1{}_d{}^e\, h^1{}_e{}^b - 12\, h^1{}_c{}^d\, h^1{}_d{}^b\, h^{2\,ac} + 6\, h^{2\,ac}\, h^2{}_c{}^b - 12\, h^{1\,ac}\, h^1{}_d{}^b\, h^2{}_c{}^d - 12\, h^{1\,ac}\, h^1{}_c{}^d\, h^2{}_d{}^b + 4\, h^1{}_c{}^b\, h^{3\,ac} + 4\, h^{1\,ac}\, h^3{}_c{}^b - h^{4\,ab}$

or the perturbations of the Ricci tensor are

```
In[47]:= Perturbation[RicciCD[-a, -b]] // ExpandPerturbation
```

Out[47]= $\frac{1}{2}\,(-h^{1\,c}{}_{c;b;a} - h^{1\,c}{}_{b;c;a} + h^1{}_{bc}{}^{;c}{}_{;a}) + \frac{1}{2}\,(h^{1\,c}{}_{b;a;c} + h^{1\,c}{}_{a;b;c} - h^1{}_{ba}{}^{;c}{}_{;c})$

```
In[48]:= Perturbation[RicciCD[-a, -b], 2] // ExpandPerturbation
```

Out[48]= $\frac{1}{2}\,(-h^{2\,c}{}_{c;b;a} - h^{2\,c}{}_{b;c;a} + h^2{}_{bc}{}^{;c}{}_{;a}) + \frac{1}{2}\,(h^{2\,c}{}_{b;a;c} + h^{2\,c}{}_{a;b;c} - h^2{}_{ba}{}^{;c}{}_{;c}) + 2\left(\frac{1}{2}\, h^{1\,cd}\,(h^1{}_{dc;b;a} + h^1{}_{db;c;a} - h^1{}_{bc;d;a}) + \frac{1}{4}\,(h^1{}_{ec;b} + h^1{}_{eb;c} - h^1{}_{cb;e})\,(h^{1\,ec}{}_{;a} + h^{1\,e}{}_a{}^{;c} - h^{1\,c}{}_a{}^{;e})\right) - 2\left(\frac{1}{2}\, h^{1\,cd}\,(h^1{}_{db;a;c} + h^1{}_{da;b;c} - h^1{}_{ba;d;c}) + \frac{1}{4}\,(h^1{}_{eb;a} + h^1{}_{ea;b} - h^1{}_{ab;e})\,(h^{1\,ec}{}_{;c} + h^{1\,e}{}_c{}^{;c} - h^{1\,c}{}_c{}^{;e})\right)$

The ExpandPerturbation command has been only predefined on certain tensors, and only when those have indices with their expected characters (for the Ricci tensor, for example, both indices are expected to be covariant; all characters are accepted for the metric). All other cases and derivatives of the tensors are handled through optional internal algorithms.

ExpandPerturbation has two options, which are by default switched on:

```
In[49]:= Options[ExpandPerturbation]

Out[49]= {SeparateMetric → True, OverDerivatives → True}
```

SeparateMetric moves indices into their character (covariant or contravariant) specified at definition time, introducing metric factors if required (this is performed using the SeparateMetric command in xTensor', hence the name of the option). For instance:

```
In[50]:= Perturbation[RicciCD[a, b]]
```

Out[50]= $\triangle[R^{ab}]$

```
In[51]:= % // ExpandPerturbation
```

Out[51]= $-g^{bd} h^{1\,ac} R_{cd} - g^{ac} h^{1\,bd} R_{cd} + g^{ac} g^{bd} \left(\frac{1}{2} (-h^{1\,e}{}_{e;d;c} - h^{1\,e}{}_{d;e;c} + h^{1}{}_{de}{}^{;e}{}_{;c}) + \frac{1}{2} (h^{1\,e}{}_{d;c;e} + h^{1\,e}{}_{c;d;e} - h^{1}{}_{dc}{}^{;e}{}_{;e}) \right)$

The additional metric factors can be absorbed using ContractMetric:

```
In[52]:= % // ContractMetric
```

Out[52]= $-h^{1\,bc} R^{a}{}_{c} - h^{1\,ac} R_{c}{}^{b} - \frac{h^{1\,c}{}_{c}{}^{;b;a}}{2} - \frac{h^{1\,cb}{}_{;c}{}^{;a}}{2} + \frac{h^{1\,b}{}_{c}{}^{;c;a}}{2} + \frac{h^{1\,cb;a}{}_{;c}}{2} + \frac{h^{1\,ca;b}{}_{;c}}{2} - \frac{h^{1\,ba;c}{}_{;c}}{2}$

```
In[53]:= ExpandPerturbation[%%%, SeparateMetric → False]
```

Out[53]= $\triangle[R^{ab}]$

OverDerivatives allows to jump over Lie or covariant derivatives. The latter gives perturbations of the Christoffel symbols, that are also expanded into derivatives of metric perturbations. For example,

```
In[54]:= Perturbation [CD[-a][RicciCD[-b, -c]]]
```

Out[54]= $\triangle[R_{bc;a}]$

```
In[55]:= % // ExpandPerturbation
```

Out[55]= $\frac{1}{2} (-h^{1\,d}{}_{d;c;b;a} - h^{1\,d}{}_{c;d;b;a} + h^{1}{}_{cd}{}^{;d}{}_{;b;a}) + \frac{1}{2} (h^{1\,d}{}_{c;b;d;a} + h^{1\,d}{}_{b;c;d;a} - h^{1}{}_{cb}{}^{;d}{}_{;d;a}) - \frac{1}{2} R_{dc} (h^{1\,d}{}_{b;a} + h^{1\,d}{}_{a;b} - h^{1}{}_{ab}{}^{;d}) - \frac{1}{2} R_{bd} (h^{1\,d}{}_{c;a} + h^{1\,d}{}_{a;c} - h^{1}{}_{ac}{}^{;d})$

```
In[56]:= ExpandPerturbation[%%, OverDerivatives → False]
```

Out[56]= $\triangle[R_{bc;a}]$

In order to show an example with Lie derivatives, we need a vector−field

```
In[57]:= DefTensor[v[a], M]

         ** DefTensor: Defining tensor v[a].
```

We perturb now at fourth−order the Lie derivative of the vector field over itself

```
In[58]:= Perturbation[LieD[v[a]][v[b]], 4]
```

Out[58]= $\Delta^4[\mathcal{L}_v v^b]$

Note that the perturbation of a Lie derivative can be written in terms of Lie derivatives only

```
In[59]:= % // ExpandPerturbation
```

Out[59]= $4\,(\mathcal{L}_{\Delta[v^a]}\Delta^3[v^b]) + 6\,(\mathcal{L}_{\Delta^2[v^a]}\Delta^2[v^b]) + 4\,(\mathcal{L}_{\Delta^3[v^a]}\Delta[v^b]) + \mathcal{L}_{\Delta^4[v^a]} v^b + \mathcal{L}_v \Delta^4[v^b]$

Again, nothing happens when switching−off the option `OverDerivatives`

```
In[60]:= ExpandPerturbation[%%, OverDerivatives → False]
```

Out[60]= $\Delta^4[\mathcal{L}_v v^b]$

3. Perturbations of curvature tensors

In this section we show how `ExpandPerturbation` acts on the most important geometric tensors of a Riemannian manifold, at any perturbative order. We shall compare its direct action at a particular order with a recursive computation of that order from lower orders. This is both a consistency check and an example of the efficiency of the direct action.

3.1. Perturbations of the inverse metric

Let us start comparing the seventh−order recursive and direct perturbations of the inverse of our metric tensor field.

Recursively :

```
In[61]:= Perturbation[g[a, b]] // ExpandPerturbation
```

Out[61]= $-h^{1\,ab}$

```
In[62]:= % // Perturbation // ExpandPerturbation
```

Out[62]= $g^{bd}\,h^{1\,ac}\,h^1{}_{cd} + g^{ac}\,h^{1\,bd}\,h^1{}_{cd} - g^{ac}\,g^{bd}\,h^2{}_{cd}$

```
In[63]:= % // Perturbation // ExpandPerturbation // org
```

Out[63]= $-6\,h^{1\,ac}\,h^{1\,bd}\,h^1{}_{cd} + 3\,h^{1\,b}{}_c\,h^{2\,ac} + 3\,h^{1\,ac}\,h^{2\,b}{}_c - h^{3\,ab}$

In[64]:= **% // Perturbation // ExpandPerturbation // org**

0.204013 Second

Out[64]=
$$24\, h^{1\,ac}\, h^{1\,bd}\, h^{1}{}_{c}{}^{e}\, h^{1}{}_{de} - 12\, h^{1\,bd}\, h^{1}{}_{cd}\, h^{2\,ac} + 6\, h^{2\,ac}\, h^{2\,b}{}_{c} -$$
$$12\, h^{1\,ac}\, h^{1}{}_{cd}\, h^{2\,bd} - 12\, h^{1\,ac}\, h^{1\,bd}\, h^{2}{}_{cd} + 4\, h^{1\,b}{}_{c}\, h^{3\,ac} + 4\, h^{1\,ac}\, h^{3\,b}{}_{c} - h^{4\,ab}$$

In[65]:= **% // Perturbation // ExpandPerturbation // org**

0.488031 Second

Out[65]=
$$-120\, h^{1\,ac}\, h^{1\,bd}\, h^{1}{}_{c}{}^{e}\, h^{1}{}_{d}{}^{f}\, h^{1}{}_{ef} + 60\, h^{1\,bd}\, h^{1}{}_{c}{}^{e}\, h^{1}{}_{de}\, h^{2\,ac} + 60\, h^{1\,ac}\, h^{1}{}_{c}{}^{e}\, h^{1}{}_{de}\, h^{2\,bd} -$$
$$30\, h^{1}{}_{cd}\, h^{2\,ac}\, h^{2\,bd} - 30\, h^{1\,bd}\, h^{2\,ac}\, h^{2}{}_{cd} - 30\, h^{1\,ac}\, h^{2\,bd}\, h^{2}{}_{cd} + 60\, h^{1\,ac}\, h^{1\,bd}\, h^{1}{}_{d}{}^{e}\, h^{2}{}_{ce} +$$
$$60\, h^{1\,ac}\, h^{1\,bd}\, h^{1}{}_{c}{}^{e}\, h^{2}{}_{de} - 20\, h^{1\,bd}\, h^{1}{}_{cd}\, h^{3\,ac} + 10\, h^{2\,b}{}_{c}\, h^{3\,ac} + 10\, h^{2\,ac}\, h^{3\,b}{}_{c} -$$
$$20\, h^{1\,ac}\, h^{1}{}_{cd}\, h^{3\,bd} - 20\, h^{1\,ac}\, h^{1\,bd}\, h^{3}{}_{cd} + 5\, h^{1\,b}{}_{c}\, h^{4\,ac} + 5\, h^{1\,ac}\, h^{4\,b}{}_{c} - h^{5\,ab}$$

In[66]:= **% // Perturbation // ExpandPerturbation // org**

1.28408 Second

Out[66]=
$$720\, h^{1\,ac}\, h^{1\,bd}\, h^{1}{}_{c}{}^{e}\, h^{1}{}_{d}{}^{f}\, h^{1}{}_{e}{}^{f1}\, h^{1}{}_{ff1} - 360\, h^{1\,bd}\, h^{1}{}_{c}{}^{e}\, h^{1}{}_{d}{}^{f}\, h^{1}{}_{ef}\, h^{2\,ac} - 360\, h^{1\,ac}\, h^{1}{}_{c}{}^{e}\, h^{1}{}_{d}{}^{f}\, h^{1}{}_{ef}\, h^{2\,bd} +$$
$$180\, h^{1}{}_{c}{}^{e}\, h^{1}{}_{de}\, h^{2\,ac}\, h^{2\,bd} - 90\, h^{2\,ac}\, h^{2\,bd}\, h^{2}{}_{cd} + 180\, h^{1\,bd}\, h^{1}{}_{d}{}^{e}\, h^{2\,ac}\, h^{2}{}_{ce} + 180\, h^{1\,ac}\, h^{1}{}_{d}{}^{e}\, h^{2\,bd}\, h^{2}{}_{ce} -$$
$$360\, h^{1\,ac}\, h^{1\,bd}\, h^{1}{}_{d}{}^{e}\, h^{1}{}_{e}{}^{f}\, h^{2}{}_{cf} + 180\, h^{1\,bd}\, h^{1}{}_{c}{}^{e}\, h^{2\,ac}\, h^{2}{}_{de} + 180\, h^{1\,ac}\, h^{1}{}_{c}{}^{e}\, h^{2\,bd}\, h^{2}{}_{de} +$$
$$180\, h^{1\,ac}\, h^{1\,bd}\, h^{2}{}_{c}{}^{e}\, h^{2}{}_{de} - 360\, h^{1\,ac}\, h^{1\,bd}\, h^{1}{}_{c}{}^{e}\, h^{1}{}_{e}{}^{f}\, h^{2}{}_{df} - 360\, h^{1\,ac}\, h^{1\,bd}\, h^{1}{}_{c}{}^{e}\, h^{1}{}_{d}{}^{f}\, h^{2}{}_{ef} +$$
$$120\, h^{1\,bd}\, h^{1}{}_{c}{}^{e}\, h^{1}{}_{de}\, h^{3\,ac} - 60\, h^{1}{}_{cd}\, h^{2\,bd}\, h^{3\,ac} - 60\, h^{1\,bd}\, h^{2}{}_{cd}\, h^{3\,ac} + 20\, h^{3\,ac}\, h^{3\,b}{}_{c} +$$
$$120\, h^{1\,ac}\, h^{1}{}_{c}{}^{e}\, h^{1}{}_{de}\, h^{3\,bd} - 60\, h^{1}{}_{cd}\, h^{2\,ac}\, h^{3\,bd} - 60\, h^{1\,ac}\, h^{2}{}_{cd}\, h^{3\,bd} - 60\, h^{1\,bd}\, h^{2\,ac}\, h^{3}{}_{cd} -$$
$$60\, h^{1\,ac}\, h^{2\,bd}\, h^{3}{}_{cd} + 120\, h^{1\,ac}\, h^{1\,bd}\, h^{1}{}_{d}{}^{e}\, h^{3}{}_{ce} + 120\, h^{1\,ac}\, h^{1\,bd}\, h^{1}{}_{c}{}^{e}\, h^{3}{}_{de} - 30\, h^{1\,bd}\, h^{1}{}_{cd}\, h^{4\,ac} +$$
$$15\, h^{2\,b}{}_{c}\, h^{4\,ac} + 15\, h^{2\,ac}\, h^{4\,b}{}_{c} - 30\, h^{1\,ac}\, h^{1}{}_{cd}\, h^{4\,bd} - 30\, h^{1\,ac}\, h^{1\,bd}\, h^{4}{}_{cd} + 6\, h^{1\,b}{}_{c}\, h^{5\,ac} + 6\, h^{1\,ac}\, h^{5\,b}{}_{c} - h^{6\,ab}$$

In[67]:= **% // Perturbation // ExpandPerturbation // org**

4.05625 Second

Out[67]=
$$-5040\, h^{1\,ac}\, h^{1\,bd}\, h^{1}{}_{c}{}^{e}\, h^{1}{}_{d}{}^{f}\, h^{1}{}_{e}{}^{f1}\, h^{1}{}_{f}{}^{f2}\, h^{1}{}_{f1f2} + 2520\, h^{1\,bd}\, h^{1}{}_{c}{}^{e}\, h^{1}{}_{d}{}^{f}\, h^{1}{}_{e}{}^{f1}\, h^{1}{}_{ff1}\, h^{2\,ac} +$$
$$2520\, h^{1\,ac}\, h^{1}{}_{c}{}^{e}\, h^{1}{}_{d}{}^{f}\, h^{1}{}_{e}{}^{f1}\, h^{1}{}_{ff1}\, h^{2\,bd} - 1260\, h^{1}{}_{c}{}^{e}\, h^{1}{}_{d}{}^{f}\, h^{1}{}_{ef}\, h^{2\,ac}\, h^{2\,bd} + 630\, h^{1}{}_{d}{}^{e}\, h^{2\,ac}\, h^{2\,bd}\, h^{2}{}_{ce} -$$
$$1260\, h^{1\,bd}\, h^{1}{}_{d}{}^{e}\, h^{1}{}_{e}{}^{f}\, h^{2\,ac}\, h^{2}{}_{cf} - 1260\, h^{1\,ac}\, h^{1}{}_{d}{}^{e}\, h^{1}{}_{e}{}^{f}\, h^{2\,bd}\, h^{2}{}_{cf} + 2520\, h^{1\,ac}\, h^{1\,bd}\, h^{1}{}_{d}{}^{e}\, h^{1}{}_{e}{}^{f}\, h^{1}{}_{f}{}^{f1}\, h^{2}{}_{cf1} +$$
$$630\, h^{1}{}_{c}{}^{e}\, h^{2\,ac}\, h^{2\,bd}\, h^{2}{}_{de} + 630\, h^{1\,bd}\, h^{2\,ac}\, h^{2}{}_{c}{}^{e}\, h^{2}{}_{de} + 630\, h^{1\,ac}\, h^{2\,bd}\, h^{2}{}_{c}{}^{e}\, h^{2}{}_{de} -$$
$$1260\, h^{1\,bd}\, h^{1}{}_{c}{}^{e}\, h^{1}{}_{e}{}^{f}\, h^{2\,ac}\, h^{2}{}_{df} - 1260\, h^{1\,ac}\, h^{1}{}_{c}{}^{e}\, h^{1}{}_{e}{}^{f}\, h^{2\,bd}\, h^{2}{}_{df} - 1260\, h^{1\,ac}\, h^{1\,bd}\, h^{1\,ef}\, h^{2}{}_{ce}\, h^{2}{}_{df} +$$
$$2520\, h^{1\,ac}\, h^{1\,bd}\, h^{1}{}_{c}{}^{e}\, h^{1}{}_{e}{}^{f}\, h^{1}{}_{f}{}^{f1}\, h^{2}{}_{df1} - 1260\, h^{1\,bd}\, h^{1}{}_{c}{}^{e}\, h^{1}{}_{d}{}^{f}\, h^{2\,ac}\, h^{2}{}_{ef} - 1260\, h^{1\,ac}\, h^{1}{}_{c}{}^{e}\, h^{1}{}_{d}{}^{f}\, h^{2\,bd}\, h^{2}{}_{ef} -$$
$$1260\, h^{1\,ac}\, h^{1\,bd}\, h^{1}{}_{d}{}^{e}\, h^{2}{}_{c}{}^{f}\, h^{2}{}_{ef} - 1260\, h^{1\,ac}\, h^{1\,bd}\, h^{1}{}_{c}{}^{e}\, h^{2}{}_{d}{}^{f}\, h^{2}{}_{ef} + 2520\, h^{1\,ac}\, h^{1\,bd}\, h^{1}{}_{c}{}^{e}\, h^{1}{}_{d}{}^{f}\, h^{1}{}_{f}{}^{f1}\, h^{2}{}_{ef1} +$$
$$2520\, h^{1\,ac}\, h^{1\,bd}\, h^{1}{}_{c}{}^{e}\, h^{1}{}_{d}{}^{f}\, h^{1}{}_{e}{}^{f1}\, h^{2}{}_{ff1} - 840\, h^{1\,bd}\, h^{1}{}_{c}{}^{e}\, h^{1}{}_{d}{}^{f}\, h^{1}{}_{ef}\, h^{3\,ac} + 420\, h^{1}{}_{c}{}^{e}\, h^{1}{}_{de}\, h^{2\,bd}\, h^{3\,ac} -$$
$$210\, h^{2\,bd}\, h^{2}{}_{cd}\, h^{3\,ac} + 420\, h^{1\,bd}\, h^{1}{}_{d}{}^{e}\, h^{2}{}_{ce}\, h^{3\,ac} + 420\, h^{1\,bd}\, h^{1}{}_{c}{}^{e}\, h^{2}{}_{de}\, h^{3\,ac} -$$
$$840\, h^{1\,ac}\, h^{1}{}_{c}{}^{e}\, h^{1}{}_{d}{}^{f}\, h^{1}{}_{ef}\, h^{3\,bd} + 420\, h^{1}{}_{c}{}^{e}\, h^{1}{}_{de}\, h^{2\,ac}\, h^{3\,bd} - 210\, h^{2\,ac}\, h^{2}{}_{cd}\, h^{3\,bd} +$$
$$420\, h^{1\,ac}\, h^{1}{}_{d}{}^{e}\, h^{2}{}_{ce}\, h^{3\,bd} + 420\, h^{1\,ac}\, h^{1}{}_{c}{}^{e}\, h^{2}{}_{de}\, h^{3\,bd} - 140\, h^{1}{}_{cd}\, h^{3\,ac}\, h^{3\,bd} - 210\, h^{2\,ac}\, h^{2\,bd}\, h^{3}{}_{cd} -$$
$$140\, h^{1\,bd}\, h^{3\,ac}\, h^{3}{}_{cd} - 140\, h^{1\,ac}\, h^{3\,bd}\, h^{3}{}_{cd} + 420\, h^{1\,bd}\, h^{1}{}_{d}{}^{e}\, h^{2\,ac}\, h^{3}{}_{ce} + 420\, h^{1\,ac}\, h^{1}{}_{d}{}^{e}\, h^{2\,bd}\, h^{3}{}_{ce} +$$
$$420\, h^{1\,ac}\, h^{1\,bd}\, h^{2}{}_{d}{}^{e}\, h^{3}{}_{ce} - 840\, h^{1\,ac}\, h^{1\,bd}\, h^{1}{}_{d}{}^{e}\, h^{1}{}_{e}{}^{f}\, h^{3}{}_{cf} + 420\, h^{1\,bd}\, h^{1}{}_{c}{}^{e}\, h^{2\,ac}\, h^{3}{}_{de} +$$
$$420\, h^{1\,ac}\, h^{1}{}_{c}{}^{e}\, h^{2\,bd}\, h^{3}{}_{de} + 420\, h^{1\,ac}\, h^{1\,bd}\, h^{2}{}_{c}{}^{e}\, h^{3}{}_{de} - 840\, h^{1\,ac}\, h^{1\,bd}\, h^{1}{}_{c}{}^{e}\, h^{1}{}_{e}{}^{f}\, h^{3}{}_{df} -$$
$$840\, h^{1\,ac}\, h^{1\,bd}\, h^{1}{}_{c}{}^{e}\, h^{1}{}_{d}{}^{f}\, h^{3}{}_{ef} + 210\, h^{1\,bd}\, h^{1}{}_{c}{}^{e}\, h^{1}{}_{de}\, h^{4\,ac} - 105\, h^{1}{}_{cd}\, h^{2\,bd}\, h^{4\,ac} -$$
$$105\, h^{1\,bd}\, h^{2}{}_{cd}\, h^{4\,ac} + 35\, h^{3\,b}{}_{c}\, h^{4\,ac} + 35\, h^{3\,ac}\, h^{4\,b}{}_{c} + 210\, h^{1\,ac}\, h^{1}{}_{c}{}^{e}\, h^{1}{}_{de}\, h^{4\,bd} -$$
$$105\, h^{1}{}_{cd}\, h^{2\,ac}\, h^{4\,bd} - 105\, h^{1\,ac}\, h^{2}{}_{cd}\, h^{4\,bd} - 105\, h^{1\,bd}\, h^{2\,ac}\, h^{4}{}_{cd} - 105\, h^{1\,ac}\, h^{2\,bd}\, h^{4}{}_{cd} +$$
$$210\, h^{1\,ac}\, h^{1\,bd}\, h^{1}{}_{d}{}^{e}\, h^{4}{}_{ce} + 210\, h^{1\,ac}\, h^{1\,bd}\, h^{1}{}_{c}{}^{e}\, h^{4}{}_{de} - 42\, h^{1\,bd}\, h^{1}{}_{cd}\, h^{5\,ac} + 21\, h^{2\,b}{}_{c}\, h^{5\,ac} +$$
$$21\, h^{2\,ac}\, h^{5\,b}{}_{c} - 42\, h^{1\,ac}\, h^{1}{}_{cd}\, h^{5\,bd} - 42\, h^{1\,ac}\, h^{1\,bd}\, h^{5}{}_{cd} + 7\, h^{1\,b}{}_{c}\, h^{6\,ac} + 7\, h^{1\,ac}\, h^{6\,b}{}_{c} - h^{7\,ab}$$

Applying `ExpandPerturbation` directly,

```
In[68]:= Perturbation[g[a, b], 7] // ExpandPerturbation
```

Out[68]=

$$\begin{aligned}
&-5040\, h^{1\,ac} h^{1}{}_{c}{}^{d} h^{1}{}_{d}{}^{e} h^{1}{}_{e}{}^{f} h^{1}{}_{f}{}^{f1} h^{1}{}_{f1}{}^{f2} h^{1}{}_{f2}{}^{b} + 2520\, h^{1}{}_{c}{}^{d} h^{1}{}_{d}{}^{e} h^{1}{}_{e}{}^{f} h^{1}{}_{f}{}^{f1} h^{1}{}_{f1}{}^{b} h^{2\,ac} + \\
&2520\, h^{1\,ac} h^{1}{}_{d}{}^{e} h^{1}{}_{e}{}^{f} h^{1}{}_{f}{}^{f1} h^{1}{}_{f1}{}^{b} h^{2}{}_{c}{}^{d} - 1260\, h^{1}{}_{d}{}^{e} h^{1}{}_{e}{}^{f} h^{1}{}_{f}{}^{b} h^{2\,ac} h^{2}{}_{c}{}^{d} + \\
&2520\, h^{1\,ac} h^{1}{}_{c}{}^{d} h^{1}{}_{e}{}^{f} h^{1}{}_{f}{}^{f1} h^{1}{}_{f1}{}^{b} h^{2}{}_{d}{}^{e} - 1260\, h^{1}{}_{c}{}^{d} h^{1}{}_{e}{}^{f} h^{1}{}_{f}{}^{b} h^{2\,ac} h^{2}{}_{d}{}^{e} - 1260\, h^{1\,ac} h^{1}{}_{e}{}^{f} h^{1}{}_{f}{}^{b} h^{2}{}_{c}{}^{d} h^{2}{}_{d}{}^{e} + \\
&630\, h^{1}{}_{e}{}^{b} h^{2\,ac} h^{2}{}_{c}{}^{d} h^{2}{}_{d}{}^{e} + 630\, h^{1}{}_{d}{}^{e} h^{2\,ac} h^{2}{}_{c}{}^{d} h^{2}{}_{e}{}^{b} + 630\, h^{1}{}_{c}{}^{d} h^{2\,ac} h^{2}{}_{d}{}^{e} h^{2}{}_{e}{}^{b} + 630\, h^{1\,ac} h^{2}{}_{c}{}^{d} h^{2}{}_{d}{}^{e} h^{2}{}_{e}{}^{b} + \\
&2520\, h^{1\,ac} h^{1}{}_{c}{}^{d} h^{1}{}_{d}{}^{e} h^{1}{}_{f}{}^{f1} h^{1}{}_{f1}{}^{b} h^{2}{}_{e}{}^{f} - 1260\, h^{1}{}_{c}{}^{d} h^{1}{}_{d}{}^{e} h^{1}{}_{f}{}^{b} h^{2\,ac} h^{2}{}_{e}{}^{f} - 1260\, h^{1\,ac} h^{1}{}_{d}{}^{e} h^{1}{}_{f}{}^{b} h^{2}{}_{c}{}^{d} h^{2}{}_{e}{}^{f} - \\
&1260\, h^{1\,ac} h^{1}{}_{c}{}^{d} h^{1}{}_{f}{}^{b} h^{2}{}_{d}{}^{e} h^{2}{}_{e}{}^{f} - 1260\, h^{1}{}_{c}{}^{d} h^{1}{}_{d}{}^{e} h^{1}{}_{e}{}^{f} h^{2\,ac} h^{2}{}_{f}{}^{b} - 1260\, h^{1\,ac} h^{1}{}_{d}{}^{e} h^{1}{}_{e}{}^{f} h^{2}{}_{c}{}^{d} h^{2}{}_{f}{}^{b} - \\
&1260\, h^{1\,ac} h^{1}{}_{c}{}^{d} h^{1}{}_{e}{}^{f} h^{2}{}_{d}{}^{e} h^{2}{}_{f}{}^{b} - 1260\, h^{1\,ac} h^{1}{}_{c}{}^{d} h^{1}{}_{d}{}^{e} h^{2}{}_{e}{}^{f} h^{2}{}_{f}{}^{b} + 2520\, h^{1\,ac} h^{1}{}_{c}{}^{d} h^{1}{}_{d}{}^{e} h^{1}{}_{e}{}^{f} h^{1}{}_{f1}{}^{b} h^{2}{}_{f}{}^{f1} + \\
&2520\, h^{1\,ac} h^{1}{}_{c}{}^{d} h^{1}{}_{d}{}^{e} h^{1}{}_{e}{}^{f} h^{1}{}_{f}{}^{f1} h^{2}{}_{f1}{}^{b} - 840\, h^{1}{}_{c}{}^{d} h^{1}{}_{d}{}^{e} h^{1}{}_{e}{}^{f} h^{1}{}_{f}{}^{b} h^{3\,ac} + 420\, h^{1}{}_{d}{}^{e} h^{1}{}_{e}{}^{b} h^{2}{}_{c}{}^{d} h^{3\,ac} - \\
&210\, h^{2}{}_{c}{}^{d} h^{2}{}_{d}{}^{b} h^{3\,ac} + 420\, h^{1}{}_{c}{}^{d} h^{1}{}_{e}{}^{b} h^{2}{}_{d}{}^{e} h^{3\,ac} + 420\, h^{1}{}_{c}{}^{d} h^{1}{}_{d}{}^{e} h^{2}{}_{e}{}^{b} h^{3\,ac} - 840\, h^{1\,ac} h^{1}{}_{d}{}^{e} h^{1}{}_{e}{}^{f} h^{1}{}_{f}{}^{b} h^{3}{}_{c}{}^{d} + \\
&420\, h^{1}{}_{d}{}^{e} h^{1}{}_{e}{}^{b} h^{2\,ac} h^{3}{}_{c}{}^{d} - 210\, h^{2\,ac} h^{2}{}_{d}{}^{b} h^{3}{}_{c}{}^{d} + 420\, h^{1\,ac} h^{1}{}_{e}{}^{b} h^{2}{}_{d}{}^{e} h^{3}{}_{c}{}^{d} + 420\, h^{1\,ac} h^{1}{}_{d}{}^{e} h^{2}{}_{e}{}^{b} h^{3}{}_{c}{}^{d} - \\
&140\, h^{1}{}_{d}{}^{b} h^{3\,ac} h^{3}{}_{c}{}^{d} - 210\, h^{2\,ac} h^{2}{}_{c}{}^{d} h^{3}{}_{d}{}^{b} - 140\, h^{1}{}_{c}{}^{d} h^{3\,ac} h^{3}{}_{d}{}^{b} - 140\, h^{1\,ac} h^{3}{}_{c}{}^{d} h^{3}{}_{d}{}^{b} - \\
&840\, h^{1\,ac} h^{1}{}_{c}{}^{d} h^{1}{}_{e}{}^{f} h^{1}{}_{f}{}^{b} h^{3}{}_{d}{}^{e} + 420\, h^{1}{}_{c}{}^{d} h^{1}{}_{e}{}^{b} h^{2\,ac} h^{3}{}_{d}{}^{e} + 420\, h^{1\,ac} h^{1}{}_{e}{}^{b} h^{2}{}_{c}{}^{d} h^{3}{}_{d}{}^{e} + \\
&420\, h^{1\,ac} h^{1}{}_{c}{}^{d} h^{2}{}_{e}{}^{b} h^{3}{}_{d}{}^{e} + 420\, h^{1}{}_{c}{}^{d} h^{1}{}_{d}{}^{e} h^{2\,ac} h^{3}{}_{e}{}^{b} + 420\, h^{1\,ac} h^{1}{}_{d}{}^{e} h^{2}{}_{c}{}^{d} h^{3}{}_{e}{}^{b} + 420\, h^{1\,ac} h^{1}{}_{c}{}^{d} h^{2}{}_{d}{}^{e} h^{3}{}_{e}{}^{b} - \\
&840\, h^{1\,ac} h^{1}{}_{c}{}^{d} h^{1}{}_{d}{}^{e} h^{1}{}_{f}{}^{b} h^{3}{}_{e}{}^{f} - 840\, h^{1\,ac} h^{1}{}_{c}{}^{d} h^{1}{}_{d}{}^{e} h^{1}{}_{e}{}^{f} h^{3}{}_{f}{}^{b} + 210\, h^{1}{}_{c}{}^{d} h^{1}{}_{d}{}^{e} h^{1}{}_{e}{}^{b} h^{4\,ac} - \\
&105\, h^{1}{}_{d}{}^{b} h^{2}{}_{c}{}^{d} h^{4\,ac} - 105\, h^{1}{}_{c}{}^{d} h^{2}{}_{d}{}^{b} h^{4\,ac} + 35\, h^{3}{}_{c}{}^{b} h^{4\,ac} + 35\, h^{3\,ac} h^{4}{}_{c}{}^{b} + 210\, h^{1\,ac} h^{1}{}_{d}{}^{e} h^{1}{}_{e}{}^{b} h^{4}{}_{c}{}^{d} - \\
&105\, h^{1}{}_{d}{}^{b} h^{2\,ac} h^{4}{}_{c}{}^{d} - 105\, h^{1\,ac} h^{2}{}_{d}{}^{b} h^{4}{}_{c}{}^{d} - 105\, h^{1}{}_{c}{}^{d} h^{2\,ac} h^{4}{}_{d}{}^{b} - 105\, h^{1\,ac} h^{2}{}_{c}{}^{d} h^{4}{}_{d}{}^{b} + \\
&210\, h^{1\,ac} h^{1}{}_{c}{}^{d} h^{1}{}_{e}{}^{b} h^{4}{}_{d}{}^{e} + 210\, h^{1\,ac} h^{1}{}_{c}{}^{d} h^{1}{}_{d}{}^{e} h^{4}{}_{e}{}^{b} - 42\, h^{1}{}_{c}{}^{d} h^{1}{}_{d}{}^{b} h^{5\,ac} + 21\, h^{2}{}_{c}{}^{b} h^{5\,ac} + \\
&21\, h^{2\,ac} h^{5}{}_{c}{}^{b} - 42\, h^{1\,ac} h^{1}{}_{d}{}^{b} h^{5}{}_{c}{}^{d} - 42\, h^{1\,ac} h^{1}{}_{c}{}^{d} h^{5}{}_{d}{}^{b} + 7\, h^{1}{}_{c}{}^{b} h^{6\,ac} + 7\, h^{1\,ac} h^{6}{}_{c}{}^{b} - h^{7\,ab}
\end{aligned}$$

Comparison and check:

```
In[69]:= % - %% // ToCanonical

         1.5481 Second

Out[69]= 0
```

Another interesting check is proving that the perturbed inverse of the metric is in fact the inverse of the perturbed metric up to the correct order. For that we will use another `xPert‘` command named `Perturbed` that expands an expression into a power series up to the order provided.

The power series of the perturbed metric up to fifth order (note that we do not use the `Series` construct of *Mathematica*)

```
In[70]:= Perturbed[g[-a, -b], 5]
```

Out[70]=

$$g_{ab} + \epsilon\, h^{1}{}_{ab} + \frac{1}{2} \epsilon^2\, h^{2}{}_{ab} + \frac{1}{6} \epsilon^3\, h^{3}{}_{ab} + \frac{1}{24} \epsilon^4\, h^{4}{}_{ab} + \frac{1}{120} \epsilon^5\, h^{5}{}_{ab}$$

The corresponding expansion for its inverse

```
In[71]:= Perturbed[g[b, c], 5] // ExpandPerturbation
```

Out[71]=

$$g^{bc} - \epsilon\, h^{1\,bc} + \frac{1}{2}\epsilon^2 \left(2\, h^{1}{}_{a}{}^{c}\, h^{1\,ba} - h^{2\,bc}\right) + \frac{1}{6}\epsilon^3 \left(-6\, h^{1}{}_{a}{}^{d}\, h^{1\,ba}\, h^{1}{}_{d}{}^{c} + 3\, h^{1}{}_{f}{}^{c}\, h^{2\,bf} + 3\, h^{1\,be}\, h^{2}{}_{e}{}^{c} - h^{3\,bc}\right) +$$

$$\frac{1}{24}\epsilon^4 \left(24\, h^{1}{}_{a}{}^{d}\, h^{1\,ba}\, h^{1}{}_{d}{}^{e}\, h^{1}{}_{e}{}^{c} - 12\, h^{1}{}_{f4}{}^{f5}\, h^{1}{}_{f5}{}^{c}\, h^{2\,bf4} - 12\, h^{1\,bf}\, h^{1}{}_{f}{}^{f1}\, h^{2}{}_{f1}{}^{c} - 12\, h^{1\,bf2}\, h^{1}{}_{f3}{}^{c}\, h^{2}{}_{f2}{}^{f3} + 6\, h^{2\,bf7}\, h^{2}{}_{f7}{}^{c} + 4\, h^{1}{}_{f8}{}^{c}\, h^{3\,bf8} + 4\, h^{1\,bf6}\, h^{3}{}_{f6}{}^{c} - h^{4\,bc}\right) +$$

$$\frac{1}{120}\epsilon^5 \left(-120\, h^{1}{}_{a}{}^{d}\, h^{1\,ba}\, h^{1}{}_{d}{}^{e}\, h^{1}{}_{e}{}^{f}\, h^{1}{}_{f}{}^{c} + 60\, h^{1}{}_{f10}{}^{f11}\, h^{1}{}_{f11}{}^{f12}\, h^{1}{}_{f12}{}^{c}\, h^{2\,bf10} - 30\, h^{1\,bf15}\, h^{2}{}_{f15}{}^{f16}\, h^{2}{}_{f16}{}^{c} - 30\, h^{1}{}_{f19}{}^{f20}\, h^{2\,bf19}\, h^{2}{}_{f20}{}^{c} - 30\, h^{1}{}_{f22}{}^{c}\, h^{2\,bf21}\, h^{2}{}_{f21}{}^{f22} + 60\, h^{1\,bf1}\, h^{1}{}_{f1}{}^{f2}\, h^{1}{}_{f2}{}^{f3}\, h^{2}{}_{f3}{}^{c} + 60\, h^{1\,bf4}\, h^{1}{}_{f4}{}^{f5}\, h^{1}{}_{f6}{}^{c}\, h^{2}{}_{f5}{}^{f6} + 60\, h^{1\,bf7}\, h^{1}{}_{f8}{}^{f9}\, h^{1}{}_{f9}{}^{c}\, h^{2}{}_{f7}{}^{f8} - 20\, h^{1}{}_{f23}{}^{f24}\, h^{1}{}_{f24}{}^{c}\, h^{3\,bf23} + 10\, h^{2}{}_{f27}{}^{c}\, h^{3\,bf27} - 20\, h^{1\,bf13}\, h^{1}{}_{f13}{}^{f14}\, h^{3}{}_{f14}{}^{c} - 20\, h^{1\,bf17}\, h^{1}{}_{f18}{}^{c}\, h^{3}{}_{f17}{}^{f18} + 10\, h^{2\,bf26}\, h^{3}{}_{f26}{}^{c} + 5\, h^{1}{}_{f28}{}^{c}\, h^{4\,bf28} + 5\, h^{1\,bf25}\, h^{4}{}_{f25}{}^{c} - h^{5\,bc}\right)$$

And the check

```
In[72]:= %%% // org
```

2.33215 Second

Out[72]=

$$\delta_{a}{}^{c} + \epsilon^6 \Big(-h^{1}{}_{a}{}^{b}\, h^{1}{}_{b}{}^{e}\, h^{1\,cd}\, h^{1}{}_{d}{}^{f}\, h^{1}{}_{e}{}^{f1}\, h^{1}{}_{ff1} + \frac{1}{2} h^{1}{}_{b}{}^{e}\, h^{1\,cd}\, h^{1}{}_{d}{}^{f}\, h^{1}{}_{ef}\, h^{2}{}_{a}{}^{b} - \frac{1}{4} h^{1\,cd}\, h^{1}{}_{d}{}^{e}\, h^{2}{}_{a}{}^{b}\, h^{2}{}_{be} + \frac{1}{2} h^{1}{}_{a}{}^{b}\, h^{1\,cd}\, h^{1}{}_{d}{}^{e}\, h^{1}{}_{e}{}^{f}\, h^{2}{}_{bf} + \frac{1}{2} h^{1}{}_{a}{}^{b}\, h^{1}{}_{b}{}^{e}\, h^{1}{}_{d}{}^{f}\, h^{1}{}_{ef}\, h^{2\,cd} - \frac{1}{4} h^{1}{}_{b}{}^{e}\, h^{1}{}_{de}\, h^{2}{}_{a}{}^{b}\, h^{2\,cd} + \frac{1}{8} h^{2}{}_{a}{}^{b}\, h^{2}{}_{bd}\, h^{2\,cd} -$$

$$\frac{1}{4} h^{1}{}_{a}{}^{b}\, h^{1}{}_{d}{}^{e}\, h^{2}{}_{be}\, h^{2\,cd} - \frac{1}{4} h^{1}{}_{b}{}^{e}\, h^{1\,cd}\, h^{2}{}_{a}{}^{b}\, h^{2}{}_{de} - \frac{1}{4} h^{1}{}_{a}{}^{b}\, h^{1\,cd}\, h^{2}{}_{b}{}^{e}\, h^{2}{}_{de} - \frac{1}{4} h^{1}{}_{a}{}^{b}\, h^{1}{}_{b}{}^{e}\, h^{2\,cd}\, h^{2}{}_{de} + \frac{1}{2} h^{1}{}_{a}{}^{b}\, h^{1}{}_{b}{}^{e}\, h^{1\,cd}\, h^{1}{}_{e}{}^{f}\, h^{2}{}_{df} + \frac{1}{2} h^{1}{}_{a}{}^{b}\, h^{1}{}_{b}{}^{e}\, h^{1\,cd}\, h^{1}{}_{d}{}^{f}\, h^{2}{}_{ef} - \frac{1}{6} h^{1}{}_{b}{}^{e}\, h^{1\,cd}\, h^{1}{}_{de}\, h^{3}{}_{a}{}^{b} +$$

$$\frac{1}{12} h^{1\,cd}\, h^{2}{}_{bd}\, h^{3}{}_{a}{}^{b} + \frac{1}{12} h^{1}{}_{bd}\, h^{2\,cd}\, h^{3}{}_{a}{}^{b} + \frac{1}{12} h^{1\,cd}\, h^{2}{}_{a}{}^{b}\, h^{3}{}_{bd} + \frac{1}{12} h^{1}{}_{a}{}^{b}\, h^{2\,cd}\, h^{3}{}_{bd} - \frac{1}{6} h^{1}{}_{a}{}^{b}\, h^{1\,cd}\, h^{1}{}_{d}{}^{e}\, h^{3}{}_{be} - \frac{1}{36} h^{3}{}_{a}{}^{b}\, h^{3\,c}{}_{b} - \frac{1}{6} h^{1}{}_{a}{}^{b}\, h^{1}{}_{b}{}^{e}\, h^{1}{}_{de}\, h^{3\,cd} + \frac{1}{12} h^{1}{}_{bd}\, h^{2}{}_{a}{}^{b}\, h^{3\,cd} +$$

$$\frac{1}{12} h^{1}{}_{a}{}^{b}\, h^{2}{}_{bd}\, h^{3\,cd} - \frac{1}{6} h^{1}{}_{a}{}^{b}\, h^{1}{}_{b}{}^{e}\, h^{1\,cd}\, h^{3}{}_{de} + \frac{1}{24} h^{1}{}_{bd}\, h^{1\,cd}\, h^{4}{}_{a}{}^{b} - \frac{1}{48} h^{2\,c}{}_{b}\, h^{4}{}_{a}{}^{b} + \frac{1}{24} h^{1}{}_{a}{}^{b}\, h^{1\,cd}\, h^{4}{}_{bd} - \frac{1}{48} h^{2}{}_{a}{}^{b}\, h^{4\,c}{}_{b} + \frac{1}{24} h^{1}{}_{a}{}^{b}\, h^{1}{}_{bd}\, h^{4\,cd} - \frac{1}{120} h^{1\,c}{}_{b}\, h^{5}{}_{a}{}^{b} - \frac{1}{120} h^{1}{}_{a}{}^{b}\, h^{5\,c}{}_{b}\Big) +$$

$$\epsilon^7 \Big(-\frac{1}{2} h^{1}{}_{b}{}^{e}\, h^{1\,cd}\, h^{1}{}_{d}{}^{f}\, h^{1}{}_{e}{}^{f1}\, h^{1}{}_{ff1}\, h^{2}{}_{a}{}^{b} + \frac{1}{4} h^{1\,cd}\, h^{1}{}_{d}{}^{e}\, h^{1}{}_{e}{}^{f}\, h^{2}{}_{a}{}^{b}\, h^{2}{}_{bf} + \frac{1}{4} h^{1}{}_{b}{}^{e}\, h^{1}{}_{d}{}^{f}\, h^{1}{}_{ef}\, h^{2}{}_{a}{}^{b}\, h^{2\,cd} - \frac{1}{8} h^{1}{}_{d}{}^{e}\, h^{2}{}_{a}{}^{b}\, h^{2}{}_{be}\, h^{2\,cd} - \frac{1}{8} h^{1\,cd}\, h^{2}{}_{a}{}^{b}\, h^{2}{}_{b}{}^{e}\, h^{2}{}_{de} - \frac{1}{8} h^{1}{}_{b}{}^{e}\, h^{2}{}_{a}{}^{b}\, h^{2\,cd}\, h^{2}{}_{de} +$$

$$\frac{1}{4} h^{1}{}_{b}{}^{e}\, h^{1\,cd}\, h^{1}{}_{e}{}^{f}\, h^{2}{}_{a}{}^{b}\, h^{2}{}_{df} + \frac{1}{4} h^{1}{}_{b}{}^{e}\, h^{1\,cd}\, h^{1}{}_{d}{}^{f}\, h^{2}{}_{a}{}^{b}\, h^{2}{}_{ef} + \frac{1}{6} h^{1}{}_{b}{}^{e}\, h^{1\,cd}\, h^{1}{}_{d}{}^{f}\, h^{1}{}_{ef}\, h^{3}{}_{a}{}^{b} - \frac{1}{12} h^{1\,cd}\, h^{1}{}_{d}{}^{e}\, h^{2}{}_{be}\, h^{3}{}_{a}{}^{b} - \frac{1}{12} h^{1}{}_{b}{}^{e}\, h^{1}{}_{de}\, h^{2\,cd}\, h^{3}{}_{a}{}^{b} + \frac{1}{24} h^{2}{}_{bd}\, h^{2\,cd}\, h^{3}{}_{a}{}^{b} - \frac{1}{12} h^{1}{}_{b}{}^{e}\, h^{1\,cd}\, h^{2}{}_{de}\, h^{3}{}_{a}{}^{b} +$$

$$\frac{1}{24} h^{2}{}_{a}{}^{b}\, h^{2\,cd}\, h^{3}{}_{bd} + \frac{1}{36} h^{1\,cd}\, h^{3}{}_{a}{}^{b}\, h^{3}{}_{bd} - \frac{1}{12} h^{1\,cd}\, h^{1}{}_{d}{}^{e}\, h^{2}{}_{a}{}^{b}\, h^{3}{}_{be} - \frac{1}{12} h^{1}{}_{b}{}^{e}\, h^{1}{}_{de}\, h^{2}{}_{a}{}^{b}\, h^{3\,cd} + \frac{1}{24} h^{2}{}_{a}{}^{b}\, h^{2}{}_{bd}\, h^{3\,cd} + \frac{1}{36} h^{1}{}_{bd}\, h^{3}{}_{a}{}^{b}\, h^{3\,cd} - \frac{1}{12} h^{1}{}_{b}{}^{e}\, h^{1\,cd}\, h^{2}{}_{a}{}^{b}\, h^{3}{}_{de} - \frac{1}{24} h^{1}{}_{b}{}^{e}\, h^{1\,cd}\, h^{1}{}_{de}\, h^{4}{}_{a}{}^{b} +$$

$$\frac{1}{48} h^{1\,cd}\, h^{2}{}_{bd}\, h^{4}{}_{a}{}^{b} + \frac{1}{48} h^{1}{}_{bd}\, h^{2\,cd}\, h^{4}{}_{a}{}^{b} - \frac{1}{144} h^{3\,c}{}_{b}\, h^{4}{}_{a}{}^{b} + \frac{1}{48} h^{1\,cd}\, h^{2}{}_{a}{}^{b}\, h^{4}{}_{bd} - \frac{1}{144} h^{3}{}_{a}{}^{b}\, h^{4\,c}{}_{b} + \frac{1}{48} h^{1}{}_{bd}\, h^{2}{}_{a}{}^{b}\, h^{4\,cd} + \frac{1}{120} h^{1}{}_{bd}\, h^{1\,cd}\, h^{5}{}_{a}{}^{b} - \frac{1}{240} h^{2\,c}{}_{b}\, h^{5}{}_{a}{}^{b} - \frac{1}{240} h^{2}{}_{a}{}^{b}\, h^{5\,c}{}_{b}\Big) +$$

$$
\begin{aligned}
&\epsilon^{8}\Big(-\frac{1}{6}h^{1}{}_{b}{}^{e}\,h^{1\,cd}\,h^{1}{}_{d}{}^{f}\,h^{1}{}_{e}{}^{f1}\,h^{1}{}_{ff1}\,h^{3}{}_{a}{}^{b}+\frac{1}{12}h^{1\,cd}\,h^{1}{}_{d}{}^{e}\,h^{1}{}_{e}{}^{f}\,h^{2}{}_{bf}\,h^{3}{}_{a}{}^{b}+\frac{1}{12}h^{1}{}_{b}{}^{e}\,h^{1}{}_{d}{}^{f}\,h^{1}{}_{ef}\,h^{2\,cd}\,h^{3}{}_{a}{}^{b}-\\
&\frac{1}{24}h^{1}{}_{d}{}^{e}\,h^{2}{}_{be}\,h^{2\,cd}\,h^{3}{}_{a}{}^{b}-\frac{1}{24}h^{1\,cd}\,h^{2}{}_{b}{}^{e}\,h^{2}{}_{de}\,h^{3}{}_{a}{}^{b}-\frac{1}{24}h^{1}{}_{b}{}^{e}\,h^{2\,cd}\,h^{2}{}_{de}\,h^{3}{}_{a}{}^{b}+\\
&\frac{1}{12}h^{1}{}_{b}{}^{e}\,h^{1\,cd}\,h^{1}{}_{e}{}^{f}\,h^{2}{}_{df}\,h^{3}{}_{a}{}^{b}+\frac{1}{12}h^{1}{}_{b}{}^{e}\,h^{1\,cd}\,h^{1}{}_{d}{}^{f}\,h^{2}{}_{ef}\,h^{3}{}_{a}{}^{b}+\frac{1}{72}h^{2\,cd}\,h^{3}{}_{a}{}^{b}\,h^{3}{}_{bd}-\\
&\frac{1}{36}h^{1\,cd}\,h^{1}{}_{d}{}^{e}\,h^{3}{}_{a}{}^{b}\,h^{3}{}_{be}-\frac{1}{36}h^{1}{}_{b}{}^{e}\,h^{1}{}_{de}\,h^{3}{}_{a}{}^{b}\,h^{3\,cd}+\frac{1}{72}h^{2}{}_{bd}\,h^{3}{}_{a}{}^{b}\,h^{3\,cd}-\frac{1}{36}h^{1}{}_{b}{}^{e}\,h^{1\,cd}\,h^{3}{}_{a}{}^{b}\,h^{3}{}_{de}+\\
&\frac{1}{24}h^{1}{}_{b}{}^{e}\,h^{1\,cd}\,h^{1}{}_{d}{}^{f}\,h^{1}{}_{ef}\,h^{4}{}_{a}{}^{b}-\frac{1}{48}h^{1\,cd}\,h^{1}{}_{d}{}^{e}\,h^{2}{}_{be}\,h^{4}{}_{a}{}^{b}-\frac{1}{48}h^{1}{}_{b}{}^{e}\,h^{1}{}_{de}\,h^{2\,cd}\,h^{4}{}_{a}{}^{b}+\\
&\frac{1}{96}h^{2}{}_{bd}\,h^{2\,cd}\,h^{4}{}_{a}{}^{b}-\frac{1}{48}h^{1}{}_{b}{}^{e}\,h^{1\,cd}\,h^{2}{}_{de}\,h^{4}{}_{a}{}^{b}+\frac{1}{144}h^{1\,cd}\,h^{3}{}_{bd}\,h^{4}{}_{a}{}^{b}+\frac{1}{144}h^{1}{}_{bd}\,h^{3\,cd}\,h^{4}{}_{a}{}^{b}+\\
&\frac{1}{144}h^{1\,cd}\,h^{3}{}_{a}{}^{b}\,h^{4}{}_{bd}-\frac{1}{576}h^{4}{}_{a}{}^{b}\,h^{4\,c}{}_{b}+\frac{1}{144}h^{1}{}_{bd}\,h^{3}{}_{a}{}^{b}\,h^{4\,cd}-\frac{1}{120}h^{1}{}_{b}{}^{e}\,h^{1\,cd}\,h^{1}{}_{de}\,h^{5}{}_{a}{}^{b}+\\
&\frac{1}{240}h^{1\,cd}\,h^{2}{}_{bd}\,h^{5}{}_{a}{}^{b}+\frac{1}{240}h^{1}{}_{bd}\,h^{2\,cd}\,h^{5}{}_{a}{}^{b}-\frac{1}{720}h^{3\,c}{}_{b}\,h^{5}{}_{a}{}^{b}-\frac{1}{720}h^{3}{}_{a}{}^{b}\,h^{5\,c}{}_{b}\Big)+\\
&\epsilon^{9}\Big(-\frac{1}{24}h^{1}{}_{b}{}^{e}\,h^{1\,cd}\,h^{1}{}_{d}{}^{f}\,h^{1}{}_{e}{}^{f1}\,h^{1}{}_{ff1}\,h^{4}{}_{a}{}^{b}+\frac{1}{48}h^{1\,cd}\,h^{1}{}_{d}{}^{e}\,h^{1}{}_{e}{}^{f}\,h^{2}{}_{bf}\,h^{4}{}_{a}{}^{b}+\\
&\frac{1}{48}h^{1}{}_{b}{}^{e}\,h^{1}{}_{d}{}^{f}\,h^{1}{}_{ef}\,h^{2\,cd}\,h^{4}{}_{a}{}^{b}-\frac{1}{96}h^{1}{}_{d}{}^{e}\,h^{2}{}_{be}\,h^{2\,cd}\,h^{4}{}_{a}{}^{b}-\frac{1}{96}h^{1\,cd}\,h^{2}{}_{b}{}^{e}\,h^{2}{}_{de}\,h^{4}{}_{a}{}^{b}-\\
&\frac{1}{96}h^{1}{}_{b}{}^{e}\,h^{2\,cd}\,h^{2}{}_{de}\,h^{4}{}_{a}{}^{b}+\frac{1}{48}h^{1}{}_{b}{}^{e}\,h^{1\,cd}\,h^{1}{}_{e}{}^{f}\,h^{2}{}_{df}\,h^{4}{}_{a}{}^{b}+\frac{1}{48}h^{1}{}_{b}{}^{e}\,h^{1\,cd}\,h^{1}{}_{d}{}^{f}\,h^{2}{}_{ef}\,h^{4}{}_{a}{}^{b}+\\
&\frac{1}{288}h^{2\,cd}\,h^{3}{}_{bd}\,h^{4}{}_{a}{}^{b}-\frac{1}{144}h^{1\,cd}\,h^{1}{}_{d}{}^{e}\,h^{3}{}_{be}\,h^{4}{}_{a}{}^{b}-\frac{1}{144}h^{1}{}_{b}{}^{e}\,h^{1}{}_{de}\,h^{3\,cd}\,h^{4}{}_{a}{}^{b}+\\
&\frac{1}{288}h^{2}{}_{bd}\,h^{3\,cd}\,h^{4}{}_{a}{}^{b}-\frac{1}{144}h^{1}{}_{b}{}^{e}\,h^{1\,cd}\,h^{3}{}_{de}\,h^{4}{}_{a}{}^{b}+\frac{1}{576}h^{1\,cd}\,h^{4}{}_{a}{}^{b}\,h^{4}{}_{bd}+\\
&\frac{1}{576}h^{1}{}_{bd}\,h^{4}{}_{a}{}^{b}\,h^{4\,cd}+\frac{1}{120}h^{1}{}_{b}{}^{e}\,h^{1\,cd}\,h^{1}{}_{d}{}^{f}\,h^{1}{}_{ef}\,h^{5}{}_{a}{}^{b}-\frac{1}{240}h^{1\,cd}\,h^{1}{}_{d}{}^{e}\,h^{2}{}_{be}\,h^{5}{}_{a}{}^{b}-\\
&\frac{1}{240}h^{1}{}_{b}{}^{e}\,h^{1}{}_{de}\,h^{2\,cd}\,h^{5}{}_{a}{}^{b}+\frac{1}{480}h^{2}{}_{bd}\,h^{2\,cd}\,h^{5}{}_{a}{}^{b}-\frac{1}{240}h^{1}{}_{b}{}^{e}\,h^{1\,cd}\,h^{2}{}_{de}\,h^{5}{}_{a}{}^{b}+\\
&\frac{1}{720}h^{1\,cd}\,h^{3}{}_{bd}\,h^{5}{}_{a}{}^{b}+\frac{1}{720}h^{1}{}_{bd}\,h^{3\,cd}\,h^{5}{}_{a}{}^{b}-\frac{h^{4\,c}{}_{b}\,h^{5}{}_{a}{}^{b}}{2880}-\frac{h^{4}{}_{a}{}^{b}\,h^{5\,c}{}_{b}}{2880}\Big)+\\
&\epsilon^{10}\Big(-\frac{1}{120}h^{1}{}_{b}{}^{e}\,h^{1\,cd}\,h^{1}{}_{d}{}^{f}\,h^{1}{}_{e}{}^{f1}\,h^{1}{}_{ff1}\,h^{5}{}_{a}{}^{b}+\frac{1}{240}h^{1\,cd}\,h^{1}{}_{d}{}^{e}\,h^{1}{}_{e}{}^{f}\,h^{2}{}_{bf}\,h^{5}{}_{a}{}^{b}+\\
&\frac{1}{240}h^{1}{}_{b}{}^{e}\,h^{1}{}_{d}{}^{f}\,h^{1}{}_{ef}\,h^{2\,cd}\,h^{5}{}_{a}{}^{b}-\frac{1}{480}h^{1}{}_{d}{}^{e}\,h^{2}{}_{be}\,h^{2\,cd}\,h^{5}{}_{a}{}^{b}-\frac{1}{480}h^{1\,cd}\,h^{2}{}_{b}{}^{e}\,h^{2}{}_{de}\,h^{5}{}_{a}{}^{b}-\\
&\frac{1}{480}h^{1}{}_{b}{}^{e}\,h^{2\,cd}\,h^{2}{}_{de}\,h^{5}{}_{a}{}^{b}+\frac{1}{240}h^{1}{}_{b}{}^{e}\,h^{1\,cd}\,h^{1}{}_{e}{}^{f}\,h^{2}{}_{df}\,h^{5}{}_{a}{}^{b}+\frac{1}{240}h^{1}{}_{b}{}^{e}\,h^{1\,cd}\,h^{1}{}_{d}{}^{f}\,h^{2}{}_{ef}\,h^{5}{}_{a}{}^{b}+\\
&\frac{h^{2\,cd}\,h^{3}{}_{bd}\,h^{5}{}_{a}{}^{b}}{1440}-\frac{1}{720}h^{1\,cd}\,h^{1}{}_{d}{}^{e}\,h^{3}{}_{be}\,h^{5}{}_{a}{}^{b}-\frac{1}{720}h^{1}{}_{b}{}^{e}\,h^{1}{}_{de}\,h^{3\,cd}\,h^{5}{}_{a}{}^{b}+\frac{h^{2}{}_{bd}\,h^{3\,cd}\,h^{5}{}_{a}{}^{b}}{1440}-\\
&\frac{1}{720}h^{1}{}_{b}{}^{e}\,h^{1\,cd}\,h^{3}{}_{de}\,h^{5}{}_{a}{}^{b}+\frac{h^{1\,cd}\,h^{4}{}_{bd}\,h^{5}{}_{a}{}^{b}}{2880}+\frac{h^{1}{}_{bd}\,h^{4\,cd}\,h^{5}{}_{a}{}^{b}}{2880}-\frac{h^{5}{}_{a}{}^{b}\,h^{5\,c}{}_{b}}{14400}\Big)
\end{aligned}
$$

3.2. Perturbations of the determinant of the metric and the antisymmetric tensor

Now we compute the perturbations of the determinant of the metric up to sixth order. Two comments are in order:

First, note that the result is always proportional to the determinant itself times a combination of contracted metric perturbations. This proportionality scalar depends on the dimension we are considering. In fact, for a given dimension N, there cannot be products of more than N metric perturbations (recall that N=4 in our examples). Timings are highly dependent on the dimensions because the internal algorithms use determinants of NxN matrices.

Second, the concept of determinant is basis−dependent (actually a density) and hence can only be treated properly using `xCoba‘`, the `xAct‘` package for component computations in a given basis. `xTensor‘` implements a fake basis called `AIndex`, which we will assume throughout this subsection.

Low order perturbations of the determinant:

```
In[73]:= Perturbation[Detg[]] // ExpandPerturbation // Simplification
```

0.180011 Second

Out[73]= $\tilde{g}\, h^{1\,a}{}_{a}$

```
In[74]:= Perturbation[Detg[], 2] // ExpandPerturbation // Simplification
```

0.408026 Second

Out[74]= $\tilde{g}\,(-h^{1}{}_{ab}\, h^{1\,ab} + h^{1\,a}{}_{a}\, h^{1\,b}{}_{b} + h^{2\,a}{}_{a})$

```
In[75]:= Perturbation[Detg[], 3] // ExpandPerturbation // Simplification
```

0.904057 Second

Out[75]= $\tilde{g}\,(2\, h^{1}{}_{a}{}^{c}\, h^{1\,ab}\, h^{1}{}_{bc} - 3\, h^{1\,ab}\, h^{2}{}_{ab} + h^{1\,a}{}_{a}\,(-3\, h^{1}{}_{bc}\, h^{1\,bc} + h^{1\,b}{}_{b}\, h^{1\,c}{}_{c} + 3\, h^{2\,b}{}_{b}) + h^{3\,a}{}_{a})$

```
In[76]:= Perturbation[Detg[], 4] // ExpandPerturbation // Simplification
```

1.95212 Second

Out[76]=
$$\tilde{g}\,(3\, h^{1}{}_{ab}\, h^{1\,ab}\, h^{1}{}_{cd}\, h^{1\,cd} - 3\, h^{2}{}_{ab}\, h^{2\,ab} - 6\, h^{1}{}_{a}{}^{c}\, h^{1\,ab}\,(h^{1}{}_{b}{}^{d}\, h^{1}{}_{cd} - 2\, h^{2}{}_{bc}) + 3\, h^{2\,a}{}_{a}\, h^{2\,b}{}_{b} - 6\, h^{1}{}_{ab}\, h^{1\,ab}\, h^{2\,c}{}_{c} - 4\, h^{1\,ab}\, h^{3}{}_{ab} + h^{1\,a}{}_{a}\,(8\, h^{1}{}_{b}{}^{d}\, h^{1\,bc}\, h^{1}{}_{cd} + h^{1\,b}{}_{b}\,(-6\, h^{1}{}_{cd}\, h^{1\,cd} + h^{1\,c}{}_{c}\, h^{1\,d}{}_{d} + 6\, h^{2\,c}{}_{c}) + 4\,(-3\, h^{1\,bc}\, h^{2}{}_{bc} + h^{3\,b}{}_{b})) + h^{4\,a}{}_{a}$$

Now the perturbative order will exceed the dimension. Still we get just products of up to four metric perturbations. In other dimen− sions these formulas would be wrong:

```
In[77]:= Perturbation[Detg[], 5] // ExpandPerturbation // Simplification
```

3.74823 Second

Out[77]=
$$\tilde{g}\,(-10\, h^{2\,ab}\, h^{3}{}_{ab} + 10\, h^{2\,a}{}_{a}\, h^{3\,b}{}_{b} + 5\, h^{1\,ab}\,(6\, h^{2}{}_{a}{}^{c}\, h^{2}{}_{bc} + 6\, h^{1}{}_{ab}\, h^{1\,cd}\, h^{2}{}_{cd} - 6\, h^{2}{}_{ab}\, h^{2\,c}{}_{c} + 4\, h^{1}{}_{a}{}^{c}\,(-3\, h^{1}{}_{b}{}^{d}\, h^{2}{}_{cd} + h^{1}{}_{bc}\, h^{2\,d}{}_{d} + h^{3}{}_{bc}) - 2\, h^{1}{}_{ab}\, h^{3\,c}{}_{c} - h^{4}{}_{ab}) - 5\, h^{1\,a}{}_{a}\,(3\, h^{2}{}_{bc}\, h^{2\,bc} - 12\, h^{1}{}_{b}{}^{d}\, h^{1\,bc}\, h^{2}{}_{cd} + 6\, h^{1\,b}{}_{b}\, h^{1\,cd}\, h^{2}{}_{cd} - 3\, h^{2\,b}{}_{b}\, h^{2\,c}{}_{c} + 6\, h^{1}{}_{bc}\, h^{1\,bc}\, h^{2\,d}{}_{d} - 2\, h^{1\,b}{}_{b}\, h^{1\,c}{}_{c}\, h^{2\,d}{}_{d} + 4\, h^{1\,bc}\, h^{3}{}_{bc} - 2\, h^{1\,b}{}_{b}\, h^{3\,c}{}_{c} - h^{4\,b}{}_{b}) + h^{5\,a}{}_{a})$$

```
In[78]:= Perturbation[Detg[], 6] // ExpandPerturbation // Simplification
```

7.05644 Second

Out[78]=
$$\tilde{g}\,(-45\, h^{2\,a}{}_{a}\, h^{2}{}_{bc}\, h^{2\,bc} + 180\, h^{1\,a}{}_{a}\, h^{1\,bc}\, h^{2}{}_{b}{}^{d}\, h^{2}{}_{cd} + 15\, h^{2\,a}{}_{a}\, h^{2\,b}{}_{b}\, h^{2\,c}{}_{c} - 45\, h^{1\,a}{}_{a}\, h^{1\,b}{}_{b}\, h^{2}{}_{cd}\, h^{2\,cd} - 180\, h^{1\,a}{}_{a}\, h^{1\,bc}\, h^{2}{}_{bc}\, h^{2\,d}{}_{d} + 45\, h^{1\,a}{}_{a}\, h^{1\,b}{}_{b}\, h^{2\,c}{}_{c}\, h^{2\,d}{}_{d} - 10\, h^{3}{}_{ab}\, h^{3\,ab} - 60\, h^{1\,a}{}_{a}\, h^{2\,bc}\, h^{3}{}_{bc} + 30\, h^{2}{}_{a}{}^{c}\,(h^{2\,ab}\, h^{2}{}_{bc} + 4\, h^{1\,ab}\, h^{3}{}_{bc}) + 10\, h^{3\,a}{}_{a}\, h^{3\,b}{}_{b} + 120\, h^{1\,a}{}_{a}\, h^{1}{}_{b}{}^{d}\, h^{1\,bc}\, h^{3}{}_{cd} - 60\, h^{1\,a}{}_{a}\, h^{1\,b}{}_{b}\, h^{1\,cd}\, h^{3}{}_{cd} + 60\, h^{1\,a}{}_{a}\, h^{2\,b}{}_{b}\, h^{3\,c}{}_{c} - 60\, h^{1\,a}{}_{a}\, h^{1}{}_{bc}\, h^{1\,bc}\, h^{3\,d}{}_{d} + 20\, h^{1\,a}{}_{a}\, h^{1\,b}{}_{b}\, h^{1\,c}{}_{c}\, h^{3\,d}{}_{d} - 15\, h^{2\,ab}\, h^{4}{}_{ab} - 30\, h^{1\,a}{}_{a}\, h^{1\,bc}\, h^{4}{}_{bc} + 15\, h^{2\,a}{}_{a}\, h^{4\,b}{}_{b} + 15\, h^{1\,a}{}_{a}\, h^{1\,b}{}_{b}\, h^{4\,c}{}_{c} - h^{1\,ab}\,(-45\, h^{1}{}_{ab}\, h^{2}{}_{cd}\, h^{2\,cd} + 45\, h^{1}{}_{ab}\, h^{2\,c}{}_{c}\, h^{2\,d}{}_{d} + 60\, h^{2\,c}{}_{c}\, h^{3}{}_{ab} + 30\, h^{1\,cd}\,(3\, h^{2}{}_{ac}\, h^{2}{}_{bd} - 3\, h^{2}{}_{ab}\, h^{2}{}_{cd} - 2\, h^{1}{}_{ab}\, h^{3}{}_{cd}) + 60\, h^{2}{}_{ab}\, h^{3\,c}{}_{c} + 10\, h^{1}{}_{a}{}^{c}\,(18\, h^{2}{}_{b}{}^{d}\, h^{2}{}_{cd} - 18\, h^{2}{}_{bc}\, h^{2\,d}{}_{d} + 12\, h^{1}{}_{b}{}^{d}\, h^{3}{}_{cd} - 4\, h^{1}{}_{bc}\, h^{3\,d}{}_{d} - 3\, h^{4}{}_{bc}) + 15\, h^{1}{}_{ab}\, h^{4\,c}{}_{c} + 6\, h^{5}{}_{ab}) + 6\, h^{1\,a}{}_{a}\, h^{5\,b}{}_{b} + h^{6\,a}{}_{a})$$

Perturbations of the antisymmetric volume−form tensor are given in terms of the perturbations of the determinant of the metric and have a similar structure.

At first−order

```
In[79]:= Perturbation[epsilong[-a, -b, -c, -d]] // ExpandPerturbation // Simplification
```

0.16801 Second

Out[79]= $\frac{1}{2}\,\epsilon g_{abcd}\,h^{1}{}_{e}{}^{e}$

At third−order (note the head `Scalar`; see the documentation of `xTensor'` for a description)

```
In[80]:= Perturbation[epsilong[-a, -b, -c, -d], 3] // ExpandPerturbation // Simplification
```

1.85212 Second

Out[80]=
$$\frac{1}{8}\,\epsilon g_{abcd}\,\Big(8\,h^{1}{}_{e}{}^{f1}\,h^{1\,ef}\,h^{1}{}_{ff1} - 12\,h^{1\,ef}\,h^{2}{}_{ef} - 2\,h^{1}{}_{e}{}^{e}\,\big(3\,h^{1}{}_{ff1}\,h^{1\,ff1} + h^{1}{}_{f}{}^{f}\,h^{1}{}^{f1}{}_{f1} - 3\,h^{2}{}_{f}{}^{f}\big) + 4\,h^{3}{}_{e}{}^{e} + 3\,\mathrm{Scalar}[h^{1}{}_{a}{}^{a}]^{3}\Big)$$

Note that now there are five `h` factors multiplying due to the perturbation of `Sqrt[Detg[]]`, and not only of `Detg[]`. The formula is still valid only in dimension 4 for the reasons above:

```
In[81]:= Perturbation[epsilong[-a, -b, -c, -d], 5] // ExpandPerturbation // Simplification
```

20.3413 Second

Out[81]=
$$\begin{aligned}
&\frac{1}{32}\,\epsilon g_{abcd}\,\Big(480\,h^{1\,ef}\,h^{2}{}_{e}{}^{f1}\,h^{2}{}_{ff1} - 960\,h^{1}{}_{e}{}^{f1}\,h^{1\,ef}\,h^{1}{}_{f}{}^{f10}\,h^{2}{}_{f1f10} - 240\,h^{1\,ef}\,h^{2}{}_{ef}\,h^{2\,f1}{}_{f1} +\\
&160\,h^{1}{}_{e}{}^{f1}\,h^{1\,ef}\,h^{1}{}_{ff1}\,h^{2\,f10}{}_{f10} - 160\,h^{2\,ef}\,h^{3}{}_{ef} + 320\,h^{1}{}_{e}{}^{f1}\,h^{1\,ef}\,h^{3}{}_{ff1} + 80\,h^{2}{}_{e}{}^{e}\,h^{3}{}_{f}{}^{f} -\\
&80\,h^{1\,ef}\,h^{4}{}_{ef} + 16\,h^{5}{}_{e}{}^{e} + 240\,h^{1}{}_{e}{}^{f1}\,h^{1\,ef}\,h^{1}{}_{ff1}\,\mathrm{Scalar}[h^{1}{}_{a}{}^{a}]^{2} - 360\,h^{1\,ef}\,h^{2}{}_{ef}\,\mathrm{Scalar}[h^{1}{}_{a}{}^{a}]^{2} +\\
&120\,h^{3}{}_{e}{}^{e}\,\mathrm{Scalar}[h^{1}{}_{a}{}^{a}]^{2} - 300\,h^{2}{}_{e}{}^{e}\,\mathrm{Scalar}[h^{1}{}_{a}{}^{a}]^{3} + 105\,\mathrm{Scalar}[h^{1}{}_{a}{}^{a}]^{5} +\\
&20\,h^{1}{}_{ef}\,h^{1\,ef}\,\big(8\,h^{1}{}_{f1}{}^{f11}\,h^{1\,f1f10}\,h^{1}{}_{f10f11} + 12\,h^{1\,f1f10}\,h^{2}{}_{f1f10} - 4\,h^{3\,f1}{}_{f1} + 15\,\mathrm{Scalar}[h^{1}{}_{a}{}^{a}]^{3}\big) +\\
&20\,h^{1}{}_{e}{}^{e}\,\Big(-18\,h^{1}{}_{ff1}\,h^{1\,ff1}\,h^{1}{}_{f10f11}\,h^{1\,f10f11} - 6\,h^{2}{}_{ff1}\,h^{2\,ff1} +\\
&\quad 12\,h^{1}{}_{f}{}^{f10}\,h^{1\,ff1}\,(h^{1}{}_{f1}{}^{f11}\,h^{1}{}_{f10f11} + 2\,h^{2}{}_{f1f10}) - 6\,h^{2}{}_{f}{}^{f}\,h^{2\,f1}{}_{f1} + 12\,h^{1}{}_{ff1}\,h^{1\,ff1}\,h^{2\,f10}{}_{f10} -\\
&\quad 8\,h^{1\,ff1}\,h^{3}{}_{ff1} + 2\,h^{4}{}_{f}{}^{f} - 18\,h^{1}{}_{ff1}\,h^{1\,ff1}\,\mathrm{Scalar}[h^{1}{}_{a}{}^{a}]^{2} + 18\,h^{2}{}_{f}{}^{f}\,\mathrm{Scalar}[h^{1}{}_{a}{}^{a}]^{2} -\\
&\quad h^{1}{}_{f}{}^{f}\,\big(24\,h^{1}{}_{f1}{}^{f11}\,h^{1\,f1f10}\,h^{1}{}_{f10f11} - 12\,h^{1\,f1f10}\,h^{2}{}_{f1f10} + 4\,h^{3\,f1}{}_{f1} + 15\,\mathrm{Scalar}[h^{1}{}_{a}{}^{a}]^{3} +\\
&\qquad h^{1\,f1}{}_{f1}\,\big(-28\,h^{1}{}_{f10f11}\,h^{1\,f10f11} + 6\,h^{1\,f10}{}_{f10}\,h^{1\,f11}{}_{f11} + 20\,h^{2\,f10}{}_{f10} - 6\,\mathrm{Scalar}[h^{1}{}_{a}{}^{a}]^{2}\big)\big) +\\
&\quad 9\,\mathrm{Scalar}[h^{1}{}_{ab}\,h^{1\,ab}]^{2} - 18\,\mathrm{Scalar}[h^{1}{}_{ab}\,h^{1\,ab}]\,\mathrm{Scalar}[h^{1}{}_{a}{}^{a}\,h^{1}{}_{b}{}^{b}] +\\
&\quad 9\,\mathrm{Scalar}[h^{1}{}_{a}{}^{a}\,h^{1}{}_{b}{}^{b}]^{2} - 18\,\mathrm{Scalar}[h^{1}{}_{ab}\,h^{1\,ab}]\,\mathrm{Scalar}[h^{2}{}_{a}{}^{a}] +\\
&\quad 18\,\mathrm{Scalar}[h^{1}{}_{a}{}^{a}\,h^{1}{}_{b}{}^{b}]\,\mathrm{Scalar}[h^{2}{}_{a}{}^{a}] + 9\,\mathrm{Scalar}[h^{2}{}_{a}{}^{a}]^{2}\Big)\Big)
\end{aligned}$$

3.3. Perturbations of the Christoffel symbols

In order to obtain the perturbation of the Christoffel symbols, we can write them in terms of partial derivatives of the metric and then apply the `ExpandPerturbation` command that will only act on metrics and inverse metrics. It is faster to do it directly applying `ExpandPerturbation`, that will calculate the perturbation by making use of non−recursive formulas for the n−th order perturbation of the Christoffel symbols. We shall check and compare the relative efficiency of both methods.

First−order perturbation of the Christoffel in terms of the metric and its inverse

```
In[82]:= Perturbation[ChristoffelCD[a, -b, -c] // ChristoffelToMetric]
```

Out[82]= $\frac{1}{2}\left(\Delta[g^{ad}]\,(g_{cd,b}+g_{bd,c}-g_{bc,d})+g^{ad}\,(h^{1}{}_{cd,b}+h^{1}{}_{bd,c}-h^{1}{}_{bc,d})\right)$

We will frequently convert partial derivatives into covariant derivatives to help canonicalizing:

```
In[83]:= CovDToChristoffel[% // ExpandPerturbation, PD, CD] // org
```

Out[83]= $-\frac{h^{1}{}_{bc}{}^{;a}}{2}+\frac{h^{1\,a}{}_{c;b}}{2}+\frac{h^{1\,a}{}_{b;c}}{2}$

Comparison with the direct action of `ExpandPerturbation`

```
In[84]:= % - (Perturbation[ChristoffelCD[a, -b, -c]] // ExpandPerturbation) // Simplification
```

Out[84]= 0

We do the same for the second−order perturbation

```
In[85]:= Perturbation[ChristoffelCD[a, -b, -c] // ChristoffelToMetric, 2]
```

Out[85]= $\frac{1}{2}\left(\Delta^{2}[g^{ad}]\,(g_{cd,b}+g_{bd,c}-g_{bc,d})+2\,\Delta[g^{ad}]\,(h^{1}{}_{cd,b}+h^{1}{}_{bd,c}-h^{1}{}_{bc,d})+g^{ad}\,(h^{2}{}_{cd,b}+h^{2}{}_{bd,c}-h^{2}{}_{bc,d})\right)$

```
In[86]:= CovDToChristoffel[% // ExpandPerturbation, PD, CD] // org
```

0.244015 Second

Out[86]= $-\frac{h^{2}{}_{bc}{}^{;a}}{2}-h^{1\,ad}\,h^{1}{}_{cd;b}+\frac{h^{2\,a}{}_{c;b}}{2}-h^{1\,ad}\,h^{1}{}_{bd;c}+\frac{h^{2\,a}{}_{b;c}}{2}+h^{1\,ad}\,h^{1}{}_{bc;d}$

```
In[87]:= % - ExpandPerturbation[Perturbation[ChristoffelCD[a, -b, -c], 2]] // ToCanonical
```

Out[87]= 0

At third order

```
In[88]:= Perturbation[ChristoffelCD[a, -b, -c] // ChristoffelToMetric, 3]
```

Out[88]= $\frac{1}{2}\left(\Delta^{3}[g^{ad}]\,(g_{cd,b}+g_{bd,c}-g_{bc,d})+3\,\Delta^{2}[g^{ad}]\,(h^{1}{}_{cd,b}+h^{1}{}_{bd,c}-h^{1}{}_{bc,d})+3\,\Delta[g^{ad}]\,(h^{2}{}_{cd,b}+h^{2}{}_{bd,c}-h^{2}{}_{bc,d})+g^{ad}\,(h^{3}{}_{cd,b}+h^{3}{}_{bd,c}-h^{3}{}_{bc,d})\right)$

```
In[89]:= CovDToChristoffel[% // ExpandPerturbation, PD, CD] // org
```

0.516032 Second

Out[89]= $-\frac{h^{3}{}_{bc}{}^{;a}}{2}-\frac{3}{2}\,h^{2\,ad}\,h^{1}{}_{cd;b}+3\,h^{1\,ad}\,h^{1}{}_{d}{}^{e}\,h^{1}{}_{ce;b}-\frac{3}{2}\,h^{1\,ad}\,h^{2}{}_{cd;b}+\frac{h^{3\,a}{}_{c;b}}{2}-\frac{3}{2}\,h^{2\,ad}\,h^{1}{}_{bd;c}+3\,h^{1\,ad}\,h^{1}{}_{d}{}^{e}\,h^{1}{}_{be;c}-\frac{3}{2}\,h^{1\,ad}\,h^{2}{}_{bd;c}+\frac{h^{3\,a}{}_{b;c}}{2}+\frac{3}{2}\,h^{2\,ad}\,h^{1}{}_{bc;d}+\frac{3}{2}\,h^{1\,ad}\,h^{2}{}_{bc;d}-3\,h^{1\,ad}\,h^{1}{}_{d}{}^{e}\,h^{1}{}_{bc;e}$

```
In[90]:= % - ExpandPerturbation[Perturbation[ChristoffelCD[a, -b, -c], 3]] // ToCanonical
```

Out[90]= 0

And finally at four order

```
In[91]:= Perturbation[ChristoffelCD[a, -b, -c] // ChristoffelToMetric, 4]
```

Out[91]=
$$\frac{1}{2}\left(\Delta^4[g^{ad}]\,(g_{cd,b}+g_{bd,c}-g_{bc,d}) + 4\,\Delta^3[g^{ad}]\,(h^1{}_{cd,b}+h^1{}_{bd,c}-h^1{}_{bc,d}) + 6\,\Delta^2[g^{ad}]\,(h^2{}_{cd,b}+h^2{}_{bd,c}-h^2{}_{bc,d}) + 4\,\Delta[g^{ad}]\,(h^3{}_{cd,b}+h^3{}_{bd,c}-h^3{}_{bc,d}) + g^{ad}\,(h^4{}_{cd,b}+h^4{}_{bd,c}-h^4{}_{bc,d})\right)$$

```
In[92]:= CovDToChristoffel[% // ExpandPerturbation, PD, CD] // org
```

1.18007 Second

Out[92]=
$$-\frac{h^4{}_{bc}{}^{;a}}{2} - 2\,h^{3\,ad}\,h^1{}_{cd;b} + 6\,h^1{}_d{}^e\,h^{2\,ad}\,h^1{}_{ce;b} + 6\,h^{1\,ad}\,h^2{}_d{}^e\,h^1{}_{ce;b} - 12\,h^{1\,ad}\,h^1{}_d{}^e\,h^1{}_e{}^f\,h^1{}_{cf;b} - 3\,h^{2\,ad}\,h^2{}_{cd;b} + 6\,h^{1\,ad}\,h^1{}_d{}^e\,h^2{}_{ce;b} - 2\,h^{1\,ad}\,h^3{}_{cd;b} + \frac{h^{4\,a}{}_{c;b}}{2} - 2\,h^{3\,ad}\,h^1{}_{bd;c} + 6\,h^1{}_d{}^e\,h^{2\,ad}\,h^1{}_{be;c} + 6\,h^{1\,ad}\,h^2{}_d{}^e\,h^1{}_{be;c} - 12\,h^{1\,ad}\,h^1{}_d{}^e\,h^1{}_e{}^f\,h^1{}_{bf;c} - 3\,h^{2\,ad}\,h^2{}_{bd;c} + 6\,h^{1\,ad}\,h^1{}_d{}^e\,h^2{}_{be;c} - 2\,h^{1\,ad}\,h^3{}_{bd;c} + \frac{h^{4\,a}{}_{b;c}}{2} + 2\,h^{3\,ad}\,h^1{}_{bc;d} + 3\,h^{2\,ad}\,h^2{}_{bc;d} + 2\,h^{1\,ad}\,h^3{}_{bc;d} - 6\,h^1{}_d{}^e\,h^{2\,ad}\,h^1{}_{bc;e} - 6\,h^{1\,ad}\,h^2{}_d{}^e\,h^1{}_{bc;e} - 6\,h^{1\,ad}\,h^1{}_d{}^e\,h^2{}_{bc;e} + 12\,h^{1\,ad}\,h^1{}_d{}^e\,h^1{}_e{}^f\,h^1{}_{bc;f}$$

```
In[93]:= % - ExpandPerturbation[Perturbation[ChristoffelCD[a, -b, -c], 4]] // ToCanonical
```

0.256015 Second

Out[93]= 0

Note that the first method is much slower: let us compute the seventh−order perturbations:

```
In[94]:= tmp1 =
           CovDToChristoffel[Perturbation[ChristoffelCD[a, -b, -c] // ChristoffelToMetric,
               7] // ExpandPerturbation, PD, CD] // org;
```

14.2929 Second

```
In[95]:= tmp2 = ExpandPerturbation[Perturbation[ChristoffelCD[a, -b, -c], 7]] // org;
```

2.37615 Second

```
In[96]:= tmp1 - tmp2 // ToCanonical
```

0.996062 Second

Out[96]= 0

3.4. Perturbations of the Riemann tensor

As we have done for the Christoffel symbols, we can calculate the perturbations of Riemann in two ways. The first one will start from the expression of Riemann tensor in terms of the metric, its inverse and partial derivatives. The second will apply directly `ExpandPerturbation`.

At first order going through the metric

```
In[97]:= Perturbation[RiemannCD[-a, -b, -c, d], 1] // RiemannToChristoffel //
          ChristoffelToMetric
```

Out[97]=
$$\frac{1}{2}\left(\triangle[g^{de}]\,(g_{ce,a,b}+g_{ae,c,b}-g_{ac,e,b})+g^{de}\,(h^{1}{}_{ce,a,b}+h^{1}{}_{ae,c,b}-h^{1}{}_{ac,e,b})+\partial_b\triangle[g^{de}]\,(g_{ce,a}+g_{ae,c}-g_{ac,e})-g^{df}\,g^{ef1}\,g_{ff1,b}\,(h^{1}{}_{ce,a}+h^{1}{}_{ae,c}-h^{1}{}_{ac,e})\right)+$$
$$\frac{1}{2}\left(-\triangle[g^{de}]\,(g_{ce,b,a}+g_{be,c,a}-g_{bc,e,a})-g^{de}\,(h^{1}{}_{ce,b,a}+h^{1}{}_{be,c,a}-h^{1}{}_{bc,e,a})-\partial_a\triangle[g^{de}]\,(g_{ce,b}+g_{be,c}-g_{bc,e})+g^{df}\,g^{ef1}\,g_{ff1,a}\,(h^{1}{}_{ce,b}+h^{1}{}_{be,c}-h^{1}{}_{bc,e})\right)+$$
$$\frac{1}{4}g^{df}\,(g_{ef,b}+g_{bf,e}-g_{be,f})\,(\triangle[g^{ef1}]\,(g_{cf1,a}+g_{af1,c}-g_{ac,f1})+g^{ef1}\,(h^{1}{}_{cf1,a}+h^{1}{}_{af1,c}-h^{1}{}_{ac,f1}))\cdot$$
$$\frac{1}{4}g^{ef}\,(g_{cf,b}+g_{bf,c}-g_{bc,f})\,(\triangle[g^{df1}]\,(g_{ef1,a}+g_{af1,e}-g_{ae,f1})+g^{df1}\,(h^{1}{}_{ef1,a}+h^{1}{}_{af1,e}-h^{1}{}_{ae,f1}))\cdot$$
$$\frac{1}{4}g^{df}\,(g_{ef,a}+g_{af,e}-g_{ae,f})\,(\triangle[g^{ef1}]\,(g_{cf1,b}+g_{bf1,c}-g_{bc,f1})+g^{ef1}\,(h^{1}{}_{cf1,b}+h^{1}{}_{bf1,c}-h^{1}{}_{bc,f1}))\cdot$$
$$\frac{1}{4}g^{ef}\,(g_{cf,a}+g_{af,c}-g_{ac,f})\,(\triangle[g^{df1}]\,(g_{ef1,b}+g_{bf1,e}-g_{be,f1})+g^{df1}\,(h^{1}{}_{ef1,b}+h^{1}{}_{bf1,e}-h^{1}{}_{be,f1}))$$

```
In[98]:= ChangeCovD[% // ExpandPerturbation, PD, CD] // org
```

5.30433 Second

Out[98]=
$$-\frac{h^{1}{}_{c}{}^{d}{}_{;b;a}}{2}-\frac{h^{1}{}_{b}{}^{d}{}_{;c;a}}{2}+\frac{h^{1}{}_{bc}{}^{;d}{}_{;a}}{2}+\frac{h^{1}{}_{c}{}^{d}{}_{;a;b}}{2}+\frac{h^{1}{}_{a}{}^{d}{}_{;c;b}}{2}-\frac{h^{1}{}_{ac}{}^{;d}{}_{;b}}{2}$$

Applying directly `ExpandPerturbation`

```
In[99]:= Perturbation[RiemannCD[-a, -b, -c, d], 1] // ExpandPerturbation
```

Out[99]=
$$\frac{1}{2}\,(-h^{1d}{}_{c;b;a}-h^{1d}{}_{b;c;a}+h^{1}{}_{cb}{}^{;d}{}_{;a})+\frac{1}{2}\,(h^{1d}{}_{c;a;b}+h^{1d}{}_{a;c;b}-h^{1}{}_{ca}{}^{;d}{}_{;b})$$

Comparison

```
In[100]:=
    %% - % // ToCanonical
```

Out[100]=
0

That result is usually presented with only four covariant derivatives, but with 10 Riemann terms (which vanish in vacuum):

```
In[101]:=
    %% // SortCovDs // ToCanonical
```

0.124008 Second

Out[101]=
$$-\frac{1}{2}h^{1de}\,R_{abce}-\frac{1}{2}h^{1}{}_{c}{}^{e}\,R_{ab}{}^{d}{}_{e}-\frac{1}{2}h^{1de}\,R_{acbe}-\frac{1}{2}h^{1}{}_{b}{}^{e}\,R_{ac}{}^{d}{}_{e}+$$
$$\frac{1}{2}h^{1}{}_{c}{}^{e}\,R_{a}{}^{d}{}_{be}+\frac{1}{2}h^{1}{}_{b}{}^{e}\,R_{a}{}^{d}{}_{ce}+\frac{1}{2}h^{1de}\,R_{aebc}-\frac{1}{2}h^{1}{}_{c}{}^{e}\,R_{aeb}{}^{d}+$$
$$\frac{1}{2}h^{1}{}_{a}{}^{e}\,R_{bc}{}^{d}{}_{e}-\frac{1}{2}h^{1}{}_{a}{}^{e}\,R_{b}{}^{d}{}_{ce}-\frac{h^{1}{}_{b}{}^{d}{}_{;a;c}}{2}+\frac{h^{1}{}_{a}{}^{d}{}_{;b;c}}{2}+\frac{h^{1}{}_{bc;a}{}^{;d}}{2}-\frac{h^{1}{}_{ac;b}{}^{;d}}{2}$$

At second order going through the metric

In[102]:=

```
Perturbation[RiemannCD[-a, -b, -c, d], 2] // RiemannToChristoffel //
 ChristoffelToMetric
```

Out[102]=

$$
\begin{aligned}
&\frac{1}{2}\Big(\triangle^2[g^{de}]\,(g_{ce,a,b}+g_{ae,c,b}-g_{ac,e,b}) + \\
&\quad g^{de}\,(h^2{}_{ce,a,b}+h^2{}_{ae,c,b}-h^2{}_{ac,e,b}) + \partial_b\,\triangle^2[g^{de}]\,(g_{ce,a}+g_{ae,c}-g_{ac,e}) + \\
&\quad 2\,(\triangle[g^{de}]\,(h^1{}_{ce,a,b}+h^1{}_{ae,c,b}-h^1{}_{ac,e,b}) + \partial_b\,\triangle[g^{de}]\,(h^1{}_{ce,a}+h^1{}_{ae,c}-h^1{}_{ac,e})) - \\
&\quad g^{df}\,g^{ef1}\,g_{ff1,b}\,(h^2{}_{ce,a}+h^2{}_{ae,c}-h^2{}_{ac,e})\Big) + \\
&\frac{1}{2}\Big(-\triangle^2[g^{de}]\,(g_{ce,b,a}+g_{be,c,a}-g_{bc,e,a}) - g^{de}\,(h^2{}_{ce,b,a}+h^2{}_{be,c,a}-h^2{}_{bc,e,a}) - \\
&\quad \partial_a\,\triangle^2[g^{de}]\,(g_{ce,b}+g_{be,c}-g_{bc,e}) - \\
&\quad 2\,(\triangle[g^{de}]\,(h^1{}_{ce,b,a}+h^1{}_{be,c,a}-h^1{}_{bc,e,a}) + \partial_a\,\triangle[g^{de}]\,(h^1{}_{ce,b}+h^1{}_{be,c}-h^1{}_{bc,e})) + \\
&\quad g^{df}\,g^{ef1}\,g_{ff1,a}\,(h^2{}_{ce,b}+h^2{}_{be,c}-h^2{}_{bc,e})\Big) + \\
&\frac{1}{2}\,(\triangle[g^{df}]\,(g_{ef,b}+g_{bf,e}-g_{be,f}) + g^{df}\,(h^1{}_{ef,b}+h^1{}_{bf,e}-h^1{}_{be,f})) \\
&\quad(\triangle[g^{ef1}]\,(g_{cf1,a}+g_{af1,c}-g_{ac,f1}) + g^{ef1}\,(h^1{}_{cf1,a}+h^1{}_{af1,c}-h^1{}_{ac,f1})) - \\
&\frac{1}{2}\,(\triangle[g^{df}]\,(g_{ef,a}+g_{af,e}-g_{ae,f}) + g^{df}\,(h^1{}_{ef,a}+h^1{}_{af,e}-h^1{}_{ae,f})) \\
&\quad(\triangle[g^{ef1}]\,(g_{cf1,b}+g_{bf1,c}-g_{bc,f1}) + g^{ef1}\,(h^1{}_{cf1,b}+h^1{}_{bf1,c}-h^1{}_{bc,f1})) + \\
&\frac{1}{4}\,g^{df}\,(g_{ef,b}+g_{bf,e}-g_{be,f})\,(\triangle^2[g^{ef1}]\,(g_{cf1,a}+g_{af1,c}-g_{ac,f1}) + \\
&\quad 2\,\triangle[g^{ef1}]\,(h^1{}_{cf1,a}+h^1{}_{af1,c}-h^1{}_{ac,f1}) + g^{ef1}\,(h^2{}_{cf1,a}+h^2{}_{af1,c}-h^2{}_{ac,f1})) - \\
&\frac{1}{4}\,g^{ef}\,(g_{cf,b}+g_{bf,c}-g_{bc,f})\,(\triangle^2[g^{df1}]\,(g_{ef1,a}+g_{af1,e}-g_{ae,f1}) + \\
&\quad 2\,\triangle[g^{df1}]\,(h^1{}_{ef1,a}+h^1{}_{af1,e}-h^1{}_{ae,f1}) + g^{df1}\,(h^2{}_{ef1,a}+h^2{}_{af1,e}-h^2{}_{ae,f1})) - \\
&\frac{1}{4}\,g^{df}\,(g_{ef,a}+g_{af,e}-g_{ae,f})\,(\triangle^2[g^{ef1}]\,(g_{cf1,b}+g_{bf1,c}-g_{bc,f1}) + \\
&\quad 2\,\triangle[g^{ef1}]\,(h^1{}_{cf1,b}+h^1{}_{bf1,c}-h^1{}_{bc,f1}) + g^{ef1}\,(h^2{}_{cf1,b}+h^2{}_{bf1,c}-h^2{}_{bc,f1})) + \\
&\frac{1}{4}\,g^{ef}\,(g_{cf,a}+g_{af,c}-g_{ac,f})\,(\triangle^2[g^{df1}]\,(g_{ef1,b}+g_{bf1,e}-g_{be,f1}) + \\
&\quad 2\,\triangle[g^{df1}]\,(h^1{}_{ef1,b}+h^1{}_{bf1,e}-h^1{}_{be,f1}) + g^{df1}\,(h^2{}_{ef1,b}+h^2{}_{bf1,e}-h^2{}_{be,f1}))
\end{aligned}
$$

In[103]:=

```
ChangeCovD[% // ExpandPerturbation, PD, CD] // org
```

19.3452 Second

Out[103]=

$$
\begin{aligned}
&h^{1\,de}\,h^1{}_{ce;b;a} - \frac{h^2{}_c{}^d{}_{;b;a}}{2} + h^{1\,de}\,h^1{}_{be;c;a} - \frac{h^2{}_b{}^d{}_{;c;a}}{2} + \frac{h^2{}_{bc}{}^{;d}{}_{;a}}{2} - h^{1\,de}\,h^1{}_{bc;e;a} + \\
&\frac{1}{2}\,h^{1\,d}{}_{e;a}\,h^1{}_c{}^e{}_{;b} - \frac{1}{2}\,h^1{}_c{}^e{}_{;a}\,h^{1\,d}{}_{e;b} - h^{1\,de}\,h^1{}_{ce;a;b} + \frac{h^2{}_c{}^d{}_{;a;b}}{2} - h^{1\,de}\,h^1{}_{ae;c;b} + \\
&\frac{h^2{}_a{}^d{}_{;c;b}}{2} - \frac{h^2{}_{ac}{}^{;d}{}_{;b}}{2} + h^{1\,de}\,h^1{}_{ac;e;b} - \frac{1}{2}\,h^{1\,d}{}_{e;b}\,h^1{}_a{}^e{}_{;c} + \frac{1}{2}\,h^{1\,d}{}_{e;a}\,h^1{}_b{}^e{}_{;c} + \\
&\frac{1}{2}\,h^1{}_{ce;b}\,h^1{}_a{}^{e;d} + \frac{1}{2}\,h^1{}_{be;c}\,h^1{}_a{}^{e;d} - \frac{1}{2}\,h^1{}_a{}^e{}_{;c}\,h^1{}_{be}{}^{;d} - \frac{1}{2}\,h^1{}_{ce;a}\,h^1{}_b{}^{e;d} - \frac{1}{2}\,h^1{}_a{}^{e;d}\,h^1{}_{bc;e} + \\
&\frac{1}{2}\,h^1{}_a{}^e{}_{;c}\,h^1{}_b{}^d{}_{;e} + \frac{1}{2}\,h^{1\,d}{}_{e;b}\,h^1{}_{ac}{}^{;e} + \frac{1}{2}\,h^1{}_{be}{}^{;d}\,h^1{}_{ac}{}^{;e} - \frac{1}{2}\,h^1{}_b{}^d{}_{;e}\,h^1{}_{ac}{}^{;e} - \\
&\frac{1}{2}\,h^1{}_{ce;b}\,h^1{}_a{}^{d;e} - \frac{1}{2}\,h^1{}_{be;c}\,h^1{}_a{}^{d;e} + \frac{1}{2}\,h^1{}_{bc;e}\,h^1{}_a{}^{d;e} - \frac{1}{2}\,h^{1\,d}{}_{e;a}\,h^1{}_{bc}{}^{;e} + \frac{1}{2}\,h^1{}_{ce;a}\,h^1{}_b{}^{d;e}
\end{aligned}
$$

Applying directly `ExpandPerturbation`

In[104]:=
```
Perturbation[RiemannCD[-a, -b, -c, d], 2] // ExpandPerturbation
```

Out[104]=

$$\frac{1}{2}\left(-h^{2\,d}{}_{c;b;a} - h^{2\,d}{}_{b;c;a} + h^{2}{}_{cb}{}^{;d}{}_{;a}\right) + \frac{1}{2}\left(h^{2\,d}{}_{c;a;b} + h^{2\,d}{}_{a;c;b} - h^{2}{}_{ca}{}^{;d}{}_{;b}\right) +$$
$$2\left(\frac{1}{2} h^{1\,de}\left(h^{1}{}_{ec;b;a} + h^{1}{}_{eb;c;a} - h^{1}{}_{cb;e;a}\right) + \frac{1}{4}\left(h^{1}{}_{fc;b} + h^{1}{}_{fb;c} - h^{1}{}_{bc;f}\right)\left(h^{1\,fd}{}_{;a} + h^{1\,f}{}_{a}{}^{;d} - h^{1\,d}{}_{a}{}^{;f}\right)\right) -$$
$$2\left(\frac{1}{2} h^{1\,de}\left(h^{1}{}_{ec;a;b} + h^{1}{}_{ea;c;b} - h^{1}{}_{ca;e;b}\right) + \frac{1}{4}\left(h^{1}{}_{fc;a} + h^{1}{}_{fa;c} - h^{1}{}_{ac;f}\right)\left(h^{1\,fd}{}_{;b} + h^{1\,f}{}_{b}{}^{;d} - h^{1\,d}{}_{b}{}^{;f}\right)\right)$$

Comparison

In[105]:=
```
%% - % // ToCanonical
```

0.512031 Second

Out[105]=
0

There is a large difference between the timings of both methods.

Here we show that the recursive method takes more of less the same time to compute the third−order perturbation as the direct method takes for the seventh−order one:

In[106]:=
```
ChangeCovD[Perturbation[RiemannCD[-a, -b, -c, d], 3] // RiemannToChristoffel //
    ChristoffelToMetric // ExpandPerturbation, PD, CD] // org;
```

57.2796 Second

In[107]:=
```
ExpandPerturbation[Perturbation[RiemannCD[-a, -b, -c, d], 7]] // org;
```

58.1196 Second

3.5. Perturbations of the Ricci tensor

The Ricci tensor is a simple trace of the Riemann tensor and hence behaves very similarly.

First−order perturbation going through the metric

In[108]:=
```
ChangeCovD[Perturbation[RiemannCD[-a, -c, -b, c] // RiemannToChristoffel //
    ChristoffelToMetric], PD, CD] // ExpandPerturbation // org
```

5.69636 Second

Out[108]=

$$-\frac{h^{1\,c}{}_{c;a;b}}{2} + \frac{h^{1}{}_{b}{}^{c}{}_{;a;c}}{2} + \frac{h^{1}{}_{a}{}^{c}{}_{;b;c}}{2} - \frac{h^{1}{}_{ab}{}^{;c}{}_{;c}}{2}$$

Applying directly `ExpandPerturbation`

In[109]:=
```
Perturbation[RicciCD[-a, -b]] // ExpandPerturbation // org
```

Out[109]=

$$-\frac{h^{1\,c}{}_{c;a;b}}{2} + \frac{h^{1}{}_{b}{}^{c}{}_{;a;c}}{2} + \frac{h^{1}{}_{a}{}^{c}{}_{;b;c}}{2} - \frac{h^{1}{}_{ab}{}^{;c}{}_{;c}}{2}$$

Comparison

In[110]:=
```
%% - % // ToCanonical
```

Out[110]=
0

Second−order perturbation going through the metric

In[111]:=
```
ChangeCovD[Perturbation[RiemannCD[-a, -c, -b, c] // RiemannToChristoffel //
    ChristoffelToMetric, 2], PD, CD] // ExpandPerturbation // org
```

20.6453 Second

Out[111]=

$$h^{1\,cd}\, h^{1}{}_{cd;b;a} + \frac{1}{2}\, h^{1\,cd}{}_{;a}\, h^{1}{}_{cd;b} - \frac{h^{2\,c}{}_{c;a;b}}{2} + \frac{1}{2}\, h^{1}{}_{b}{}^{c}{}_{;a}\, h^{1\,d}{}_{d;c} + \frac{1}{2}\, h^{1}{}_{a}{}^{c}{}_{;b}\, h^{1\,d}{}_{d;c} + \frac{h^{2}{}_{b}{}^{c}{}_{;a;c}}{2} + \frac{h^{2}{}_{a}{}^{c}{}_{;b;c}}{2} - \frac{h^{2}{}_{ab}{}^{;c}{}_{;c}}{2} - \frac{1}{2}\, h^{1\,d}{}_{d;c}\, h^{1}{}_{ab}{}^{;c} - h^{1}{}_{b}{}^{c}{}_{;a}\, h^{1}{}_{c}{}^{d}{}_{;d} - h^{1}{}_{a}{}^{c}{}_{;b}\, h^{1}{}_{c}{}^{d}{}_{;d} + h^{1}{}_{ab}{}^{;c}\, h^{1}{}_{c}{}^{d}{}_{;d} - h^{1\,cd}\, h^{1}{}_{bc;a;d} - h^{1\,cd}\, h^{1}{}_{ac;b;d} + h^{1\,cd}\, h^{1}{}_{ab;c;d} - h^{1}{}_{bd;c}\, h^{1}{}_{a}{}^{c;d} + h^{1}{}_{bc;d}\, h^{1}{}_{a}{}^{c;d}$$

Applying directly `ExpandPerturbation`

In[112]:=
```
Perturbation[RicciCD[-a, -b], 2] // ExpandPerturbation // org
```

0.240015 Second

Out[112]=

$$h^{1\,cd}\, h^{1}{}_{cd;b;a} + \frac{1}{2}\, h^{1\,cd}{}_{;a}\, h^{1}{}_{cd;b} - \frac{h^{2\,c}{}_{c;a;b}}{2} + \frac{1}{2}\, h^{1}{}_{b}{}^{c}{}_{;a}\, h^{1\,d}{}_{d;c} + \frac{1}{2}\, h^{1}{}_{a}{}^{c}{}_{;b}\, h^{1\,d}{}_{d;c} + \frac{h^{2}{}_{b}{}^{c}{}_{;a;c}}{2} + \frac{h^{2}{}_{a}{}^{c}{}_{;b;c}}{2} - \frac{h^{2}{}_{ab}{}^{;c}{}_{;c}}{2} - \frac{1}{2}\, h^{1\,d}{}_{d;c}\, h^{1}{}_{ab}{}^{;c} - h^{1}{}_{b}{}^{c}{}_{;a}\, h^{1}{}_{c}{}^{d}{}_{;d} - h^{1}{}_{a}{}^{c}{}_{;b}\, h^{1}{}_{c}{}^{d}{}_{;d} + h^{1}{}_{ab}{}^{;c}\, h^{1}{}_{c}{}^{d}{}_{;d} - h^{1\,cd}\, h^{1}{}_{bc;a;d} - h^{1\,cd}\, h^{1}{}_{ac;b;d} + h^{1\,cd}\, h^{1}{}_{ab;c;d} - h^{1}{}_{bd;c}\, h^{1}{}_{a}{}^{c;d} + h^{1}{}_{bc;d}\, h^{1}{}_{a}{}^{c;d}$$

Comparison

In[113]:=
```
%% - % // ToCanonical
```

Out[113]=
0

Third−order perturbation going through the metric

In[114]:=

```
ChangeCovD[Perturbation[RiemannCD[-a, -c, -b, c] // RiemannToChristoffel //
    ChristoffelToMetric, 3], PD, CD] // ExpandPerturbation // org
```

```
64.388 Second
```

Out[114]=

$$
\begin{aligned}
&\frac{3}{2} h^{2\,cd} h^{1}{}_{cd;b;a} - 3 h^{1}{}_{c}{}^{e} h^{1\,cd} h^{1}{}_{de;b;a} + \frac{3}{2} h^{1\,cd} h^{2}{}_{cd;b;a} + \frac{3}{4} h^{2\,cd}{}_{;a} h^{1}{}_{cd;b} - \\
&3 h^{1\,cd} h^{1}{}_{c}{}^{e}{}_{;a} h^{1}{}_{de;b} + \frac{3}{4} h^{1\,cd}{}_{;a} h^{2}{}_{cd;b} - \frac{h^{3\,c}{}_{c;a;b}}{2} + \frac{3}{4} h^{2}{}_{b}{}^{c}{}_{;a} h^{1\,d}{}_{d;c} + \frac{3}{4} h^{2}{}_{a}{}^{c}{}_{;b} h^{1\,d}{}_{d;c} + \\
&\frac{3}{4} h^{1}{}_{b}{}^{c}{}_{;a} h^{2\,d}{}_{d;c} + \frac{3}{4} h^{1}{}_{a}{}^{c}{}_{;b} h^{2\,d}{}_{d;c} + \frac{h^{3}{}_{b}{}^{c}{}_{;a;c}}{2} + \frac{h^{3}{}_{a}{}^{c}{}_{;b;c}}{2} - \frac{h^{3}{}_{ab}{}^{;c}{}_{;c}}{2} - \frac{3}{4} h^{2\,d}{}_{d;c} h^{1}{}_{ab}{}^{;c} - \\
&\frac{3}{4} h^{1\,d}{}_{d;c} h^{2}{}_{ab}{}^{;c} - 3 h^{1\,cd} h^{1}{}_{a}{}^{e}{}_{;c} h^{1}{}_{be;d} - \frac{3}{2} h^{2}{}_{b}{}^{c}{}_{;a} h^{1}{}_{c}{}^{d}{}_{;d} - \frac{3}{2} h^{2}{}_{a}{}^{c}{}_{;b} h^{1}{}_{c}{}^{d}{}_{;d} + \frac{3}{2} h^{2}{}_{ab}{}^{;c} h^{1}{}_{c}{}^{d}{}_{;d} + \\
&3 h^{1\,cd} h^{1}{}_{b}{}^{e}{}_{;a} h^{1}{}_{ce;d} + 3 h^{1\,cd} h^{1}{}_{a}{}^{e}{}_{;b} h^{1}{}_{ce;d} - \frac{3}{2} h^{1\,cd} h^{1}{}_{bc;a} h^{1\,e}{}_{e;d} - \frac{3}{2} h^{1\,cd} h^{1}{}_{ac;b} h^{1\,e}{}_{e;d} + \\
&\frac{3}{2} h^{1\,cd} h^{1}{}_{ab;c} h^{1\,e}{}_{e;d} - \frac{3}{2} h^{1}{}_{b}{}^{c}{}_{;a} h^{2}{}_{c}{}^{d}{}_{;d} - \frac{3}{2} h^{1}{}_{a}{}^{c}{}_{;b} h^{2}{}_{c}{}^{d}{}_{;d} + \frac{3}{2} h^{1}{}_{ab}{}^{;c} h^{2}{}_{c}{}^{d}{}_{;d} - \frac{3}{2} h^{2\,cd} h^{1}{}_{bc;a;d} - \\
&\frac{3}{2} h^{1\,cd} h^{2}{}_{bc;a;d} - \frac{3}{2} h^{2\,cd} h^{1}{}_{ac;b;d} - \frac{3}{2} h^{1\,cd} h^{2}{}_{ac;b;d} + \frac{3}{2} h^{2\,cd} h^{1}{}_{ab;c;d} + \frac{3}{2} h^{1\,cd} h^{2}{}_{ab;c;d} - \\
&\frac{3}{2} h^{2}{}_{bd;c} h^{1}{}_{a}{}^{c;d} + \frac{3}{2} h^{2}{}_{bc;d} h^{1}{}_{a}{}^{c;d} - \frac{3}{2} h^{1}{}_{bd;c} h^{2}{}_{a}{}^{c;d} + \frac{3}{2} h^{1}{}_{bc;d} h^{2}{}_{a}{}^{c;d} + 3 h^{1\,cd} h^{1}{}_{a}{}^{e}{}_{;c} h^{1}{}_{bd;e} - \\
&\frac{3}{2} h^{1\,cd} h^{1}{}_{b}{}^{e}{}_{;a} h^{1}{}_{cd;e} - \frac{3}{2} h^{1\,cd} h^{1}{}_{a}{}^{e}{}_{;b} h^{1}{}_{cd;e} + 3 h^{1\,cd} h^{1}{}_{bc;a} h^{1}{}_{d}{}^{e}{}_{;e} + 3 h^{1\,cd} h^{1}{}_{ac;b} h^{1}{}_{d}{}^{e}{}_{;e} - \\
&3 h^{1\,cd} h^{1}{}_{ab;c} h^{1}{}_{d}{}^{e}{}_{;e} + 3 h^{1}{}_{c}{}^{e} h^{1\,cd} h^{1}{}_{bd;a;e} + 3 h^{1}{}_{c}{}^{e} h^{1\,cd} h^{1}{}_{ad;b;e} - 3 h^{1}{}_{c}{}^{e} h^{1\,cd} h^{1}{}_{ab;d;e} - \\
&3 h^{1\,cd} h^{1}{}_{ce;d} h^{1}{}_{ab}{}^{;e} + \frac{3}{2} h^{1\,cd} h^{1}{}_{cd;e} h^{1}{}_{ab}{}^{;e} + 3 h^{1\,cd} h^{1}{}_{be;d} h^{1}{}_{ac}{}^{;e} - 3 h^{1\,cd} h^{1}{}_{bd;e} h^{1}{}_{ac}{}^{;e}
\end{aligned}
$$

Applying directly `ExpandPerturbation`

In[115]:=

```
Perturbation[RicciCD[-a, -b], 3] // ExpandPerturbation // org
```

```
0.876055 Second
```

Out[115]=

$$
\begin{aligned}
&\frac{3}{2} h^{2\,cd} h^{1}{}_{cd;b;a} - 3 h^{1}{}_{c}{}^{e} h^{1\,cd} h^{1}{}_{de;b;a} + \frac{3}{2} h^{1\,cd} h^{2}{}_{cd;b;a} + \frac{3}{4} h^{2\,cd}{}_{;a} h^{1}{}_{cd;b} - \\
&3 h^{1\,cd} h^{1}{}_{c}{}^{e}{}_{;a} h^{1}{}_{de;b} + \frac{3}{4} h^{1\,cd}{}_{;a} h^{2}{}_{cd;b} - \frac{h^{3\,c}{}_{c;a;b}}{2} + \frac{3}{4} h^{2}{}_{b}{}^{c}{}_{;a} h^{1\,d}{}_{d;c} + \frac{3}{4} h^{2}{}_{a}{}^{c}{}_{;b} h^{1\,d}{}_{d;c} + \\
&\frac{3}{4} h^{1}{}_{b}{}^{c}{}_{;a} h^{2\,d}{}_{d;c} + \frac{3}{4} h^{1}{}_{a}{}^{c}{}_{;b} h^{2\,d}{}_{d;c} + \frac{h^{3}{}_{b}{}^{c}{}_{;a;c}}{2} + \frac{h^{3}{}_{a}{}^{c}{}_{;b;c}}{2} - \frac{h^{3}{}_{ab}{}^{;c}{}_{;c}}{2} - \frac{3}{4} h^{2\,d}{}_{d;c} h^{1}{}_{ab}{}^{;c} - \\
&\frac{3}{4} h^{1\,d}{}_{d;c} h^{2}{}_{ab}{}^{;c} - 3 h^{1\,cd} h^{1}{}_{a}{}^{e}{}_{;c} h^{1}{}_{be;d} - \frac{3}{2} h^{2}{}_{b}{}^{c}{}_{;a} h^{1}{}_{c}{}^{d}{}_{;d} - \frac{3}{2} h^{2}{}_{a}{}^{c}{}_{;b} h^{1}{}_{c}{}^{d}{}_{;d} + \frac{3}{2} h^{2}{}_{ab}{}^{;c} h^{1}{}_{c}{}^{d}{}_{;d} + \\
&3 h^{1\,cd} h^{1}{}_{b}{}^{e}{}_{;a} h^{1}{}_{ce;d} + 3 h^{1\,cd} h^{1}{}_{a}{}^{e}{}_{;b} h^{1}{}_{ce;d} - \frac{3}{2} h^{1\,cd} h^{1}{}_{bc;a} h^{1\,e}{}_{e;d} - \frac{3}{2} h^{1\,cd} h^{1}{}_{ac;b} h^{1\,e}{}_{e;d} + \\
&\frac{3}{2} h^{1\,cd} h^{1}{}_{ab;c} h^{1\,e}{}_{e;d} - \frac{3}{2} h^{1}{}_{b}{}^{c}{}_{;a} h^{2}{}_{c}{}^{d}{}_{;d} - \frac{3}{2} h^{1}{}_{a}{}^{c}{}_{;b} h^{2}{}_{c}{}^{d}{}_{;d} + \frac{3}{2} h^{1}{}_{ab}{}^{;c} h^{2}{}_{c}{}^{d}{}_{;d} - \frac{3}{2} h^{2\,cd} h^{1}{}_{bc;a;d} - \\
&\frac{3}{2} h^{1\,cd} h^{2}{}_{bc;a;d} - \frac{3}{2} h^{2\,cd} h^{1}{}_{ac;b;d} - \frac{3}{2} h^{1\,cd} h^{2}{}_{ac;b;d} + \frac{3}{2} h^{2\,cd} h^{1}{}_{ab;c;d} + \frac{3}{2} h^{1\,cd} h^{2}{}_{ab;c;d} - \\
&\frac{3}{2} h^{2}{}_{bd;c} h^{1}{}_{a}{}^{c;d} + \frac{3}{2} h^{2}{}_{bc;d} h^{1}{}_{a}{}^{c;d} - \frac{3}{2} h^{1}{}_{bd;c} h^{2}{}_{a}{}^{c;d} + \frac{3}{2} h^{1}{}_{bc;d} h^{2}{}_{a}{}^{c;d} + 3 h^{1\,cd} h^{1}{}_{a}{}^{e}{}_{;c} h^{1}{}_{bd;e} - \\
&\frac{3}{2} h^{1\,cd} h^{1}{}_{b}{}^{e}{}_{;a} h^{1}{}_{cd;e} - \frac{3}{2} h^{1\,cd} h^{1}{}_{a}{}^{e}{}_{;b} h^{1}{}_{cd;e} + 3 h^{1\,cd} h^{1}{}_{bc;a} h^{1}{}_{d}{}^{e}{}_{;e} + 3 h^{1\,cd} h^{1}{}_{ac;b} h^{1}{}_{d}{}^{e}{}_{;e} - \\
&3 h^{1\,cd} h^{1}{}_{ab;c} h^{1}{}_{d}{}^{e}{}_{;e} + 3 h^{1}{}_{c}{}^{e} h^{1\,cd} h^{1}{}_{bd;a;e} + 3 h^{1}{}_{c}{}^{e} h^{1\,cd} h^{1}{}_{ad;b;e} - 3 h^{1}{}_{c}{}^{e} h^{1\,cd} h^{1}{}_{ab;d;e} - \\
&3 h^{1\,cd} h^{1}{}_{ce;d} h^{1}{}_{ab}{}^{;e} + \frac{3}{2} h^{1\,cd} h^{1}{}_{cd;e} h^{1}{}_{ab}{}^{;e} + 3 h^{1\,cd} h^{1}{}_{be;d} h^{1}{}_{ac}{}^{;e} - 3 h^{1\,cd} h^{1}{}_{bd;e} h^{1}{}_{ac}{}^{;e}
\end{aligned}
$$

Comparison

In[116]:=

```
%% - % // ToCanonical
```

```
0.200013 Second
```

Out[116]=

```
0
```

Let us just compute the fourth−order perturbation making use only of the direct approach. Note that what really takes times is not to construct the perturbation itself but to simplify it.

In[117]:=

```
Perturbation[RicciCD[-a, -b], 4] // ExpandPerturbation;
```

In[118]:=
```
% // org
```
3.1522 Second

Out[118]=

$$2 h^{3\,cd} h^{1}{}_{cd;b;a} - 12 h^{1\,cd} h^{2}{}_{c}{}^{e} h^{1}{}_{de;b;a} + 12 h^{1}{}_{c}{}^{e} h^{1\,cd} h^{1}{}_{d}{}^{f} h^{1}{}_{ef;b;a} + 3 h^{2\,cd} h^{2}{}_{cd;b;a} -$$
$$6 h^{1}{}_{c}{}^{e} h^{1\,cd} h^{2}{}_{de;b;a} + 2 h^{1\,cd} h^{3}{}_{cd;b;a} + h^{3\,cd}{}_{;a} h^{1}{}_{cd;b} - 6 h^{2\,cd} h^{1}{}_{c}{}^{e}{}_{;a} h^{1}{}_{de;b} - 6 h^{1\,cd} h^{2}{}_{c}{}^{e}{}_{;a} h^{1}{}_{de;b} +$$
$$6 h^{1\,cd} h^{1\,ef} h^{1}{}_{ce;a} h^{1}{}_{df;b} + 12 h^{1}{}_{c}{}^{e} h^{1\,cd} h^{1}{}_{d}{}^{f}{}_{;a} h^{1}{}_{ef;b} + \frac{3}{2} h^{2\,cd}{}_{;a} h^{2}{}_{cd;b} - 6 h^{1\,cd} h^{1}{}_{c}{}^{e}{}_{;a} h^{2}{}_{de;b} +$$
$$h^{1\,cd}{}_{;a} h^{3}{}_{cd;b} - \frac{h^{4\,c}{}_{c;a;b}}{2} + h^{3}{}_{b}{}^{c}{}_{;a} h^{1\,d}{}_{d;c} + h^{3}{}_{a}{}^{c}{}_{;b} h^{1\,d}{}_{d;c} + \frac{3}{2} h^{2}{}_{b}{}^{c}{}_{;a} h^{2\,d}{}_{d;c} + \frac{3}{2} h^{2}{}_{a}{}^{c}{}_{;b} h^{2\,d}{}_{d;c} +$$
$$h^{1}{}_{b}{}^{c}{}_{;a} h^{3\,d}{}_{d;c} + h^{1}{}_{a}{}^{c}{}_{;b} h^{3\,d}{}_{d;c} + \frac{h^{4}{}_{b}{}^{c}{}_{;a;c}}{2} + \frac{h^{4}{}_{a}{}^{c}{}_{;b;c}}{2} - \frac{h^{4}{}_{ab}{}^{;c}{}_{;c}}{2} - h^{3\,d}{}_{d;c} h^{1}{}_{ab}{}^{;c} - \frac{3}{2} h^{2\,d}{}_{d;c} h^{2}{}_{ab}{}^{;c} -$$
$$h^{1\,d}{}_{d;c} h^{3}{}_{ab}{}^{;c} - 6 h^{2\,cd} h^{1}{}_{a}{}^{e}{}_{;c} h^{1}{}_{be;d} - 6 h^{1\,cd} h^{2}{}_{a}{}^{e}{}_{;c} h^{1}{}_{be;d} - 2 h^{3}{}_{b}{}^{c}{}_{;a} h^{1}{}_{c}{}^{d}{}_{;d} - 2 h^{3}{}_{a}{}^{c}{}_{;b} h^{1}{}_{c}{}^{d}{}_{;d} +$$
$$2 h^{3}{}_{ab}{}^{;c} h^{1}{}_{c}{}^{d}{}_{;d} + 6 h^{2\,cd} h^{1}{}_{b}{}^{e}{}_{;a} h^{1}{}_{ce;d} + 6 h^{1\,cd} h^{2}{}_{b}{}^{e}{}_{;a} h^{1}{}_{ce;d} + 6 h^{2\,cd} h^{1}{}_{a}{}^{e}{}_{;b} h^{1}{}_{ce;d} + 6 h^{1\,cd} h^{2}{}_{a}{}^{e}{}_{;b} h^{1}{}_{ce;d} +$$
$$6 h^{1\,cd} h^{1\,ef} h^{1}{}_{bc;a} h^{1}{}_{ef;d} + 6 h^{1\,cd} h^{1\,ef} h^{1}{}_{ac;b} h^{1}{}_{ef;d} - 6 h^{1\,cd} h^{1\,ef} h^{1}{}_{ab;c} h^{1}{}_{ef;d} - 3 h^{2\,cd} h^{1}{}_{bc;a} h^{1\,e}{}_{e;d} -$$
$$3 h^{1\,cd} h^{2}{}_{bc;a} h^{1\,e}{}_{e;d} - 3 h^{2\,cd} h^{1}{}_{ac;b} h^{1\,e}{}_{e;d} - 3 h^{1\,cd} h^{2}{}_{ac;b} h^{1\,e}{}_{e;d} + 3 h^{2\,cd} h^{1}{}_{ab;c} h^{1\,e}{}_{e;d} +$$
$$3 h^{1\,cd} h^{2}{}_{ab;c} h^{1\,e}{}_{e;d} - 6 h^{1\,cd} h^{1}{}_{a}{}^{e}{}_{;c} h^{2}{}_{be;d} - 3 h^{2}{}_{b}{}^{c}{}_{;a} h^{2}{}_{c}{}^{d}{}_{;d} - 3 h^{2}{}_{a}{}^{c}{}_{;b} h^{2}{}_{c}{}^{d}{}_{;d} + 3 h^{2}{}_{ab}{}^{;c} h^{2}{}_{c}{}^{d}{}_{;d} +$$
$$6 h^{1\,cd} h^{1}{}_{b}{}^{e}{}_{;a} h^{2}{}_{ce;d} + 6 h^{1\,cd} h^{1}{}_{a}{}^{e}{}_{;b} h^{2}{}_{ce;d} - 3 h^{1\,cd} h^{1}{}_{bc;a} h^{2\,e}{}_{e;d} - 3 h^{1\,cd} h^{1}{}_{ac;b} h^{2\,e}{}_{e;d} +$$
$$3 h^{1\,cd} h^{1}{}_{ab;c} h^{2\,e}{}_{e;d} - 2 h^{1}{}_{b}{}^{c}{}_{;a} h^{3}{}_{c}{}^{d}{}_{;d} - 2 h^{1}{}_{a}{}^{c}{}_{;b} h^{3}{}_{c}{}^{d}{}_{;d} + 2 h^{1}{}_{ab}{}^{;c} h^{3}{}_{c}{}^{d}{}_{;d} - 2 h^{3\,cd} h^{1}{}_{bc;a;d} +$$
$$6 h^{1\,cd} h^{2}{}_{c}{}^{e} h^{1}{}_{be;a;d} - 3 h^{2\,cd} h^{2}{}_{bc;a;d} - 2 h^{1\,cd} h^{3}{}_{bc;a;d} - 2 h^{3\,cd} h^{1}{}_{ac;b;d} + 6 h^{1\,cd} h^{2}{}_{c}{}^{e} h^{1}{}_{ae;b;d} -$$
$$3 h^{2\,cd} h^{2}{}_{ac;b;d} - 2 h^{1\,cd} h^{3}{}_{ac;b;d} + 2 h^{3\,cd} h^{1}{}_{ab;c;d} + 3 h^{2\,cd} h^{2}{}_{ab;c;d} + 2 h^{1\,cd} h^{3}{}_{ab;c;d} -$$
$$6 h^{1\,cd} h^{2}{}_{c}{}^{e} h^{1}{}_{ab;e;d} - 2 h^{3}{}_{bd;c} h^{1}{}_{a}{}^{c;d} + 2 h^{3}{}_{bc;d} h^{1}{}_{a}{}^{c;d} - 3 h^{2}{}_{bd;c} h^{2}{}_{a}{}^{c;d} + 3 h^{2}{}_{bc;d} h^{2}{}_{a}{}^{c;d} -$$
$$2 h^{1}{}_{bd;c} h^{3}{}_{a}{}^{c;d} + 2 h^{1}{}_{bc;d} h^{3}{}_{a}{}^{c;d} - 12 h^{1\,cd} h^{1\,ef} h^{1}{}_{bf;d} h^{1}{}_{ac;e} + 6 h^{2\,cd} h^{1}{}_{a}{}^{e}{}_{;c} h^{1}{}_{bd;e} +$$
$$6 h^{1\,cd} h^{2}{}_{a}{}^{e}{}_{;c} h^{1}{}_{bd;e} + 12 h^{1}{}_{c}{}^{e} h^{1\,cd} h^{1}{}_{a}{}^{f}{}_{;d} h^{1}{}_{bf;e} - 3 h^{2\,cd} h^{1}{}_{b}{}^{e}{}_{;a} h^{1}{}_{cd;e} - 3 h^{1\,cd} h^{2}{}_{b}{}^{e}{}_{;a} h^{1}{}_{cd;e} -$$
$$3 h^{2\,cd} h^{1}{}_{a}{}^{e}{}_{;b} h^{1}{}_{cd;e} - 3 h^{1\,cd} h^{2}{}_{a}{}^{e}{}_{;b} h^{1}{}_{cd;e} + 6 h^{2\,cd} h^{1}{}_{bc;a} h^{1}{}_{d}{}^{e}{}_{;e} + 6 h^{1\,cd} h^{2}{}_{bc;a} h^{1}{}_{d}{}^{e}{}_{;e} +$$
$$6 h^{2\,cd} h^{1}{}_{ac;b} h^{1}{}_{d}{}^{e}{}_{;e} + 6 h^{1\,cd} h^{2}{}_{ac;b} h^{1}{}_{d}{}^{e}{}_{;e} - 6 h^{2\,cd} h^{1}{}_{ab;c} h^{1}{}_{d}{}^{e}{}_{;e} - 6 h^{1\,cd} h^{2}{}_{ab;c} h^{1}{}_{d}{}^{e}{}_{;e} -$$
$$12 h^{1}{}_{c}{}^{e} h^{1\,cd} h^{1}{}_{b}{}^{f}{}_{;a} h^{1}{}_{df;e} - 12 h^{1}{}_{c}{}^{e} h^{1\,cd} h^{1}{}_{a}{}^{f}{}_{;b} h^{1}{}_{df;e} + 6 h^{1}{}_{c}{}^{e} h^{1\,cd} h^{1}{}_{bd;a} h^{1\,f}{}_{f;e} +$$
$$6 h^{1}{}_{c}{}^{e} h^{1\,cd} h^{1}{}_{ad;b} h^{1\,f}{}_{f;e} - 6 h^{1}{}_{c}{}^{e} h^{1\,cd} h^{1}{}_{ab;d} h^{1\,f}{}_{f;e} + 6 h^{1\,cd} h^{1}{}_{a}{}^{e}{}_{;c} h^{2}{}_{bd;e} - 3 h^{1\,cd} h^{1}{}_{b}{}^{e}{}_{;a} h^{2}{}_{cd;e} -$$
$$3 h^{1\,cd} h^{1}{}_{a}{}^{e}{}_{;b} h^{2}{}_{cd;e} + 6 h^{1\,cd} h^{1}{}_{bc;a} h^{2}{}_{d}{}^{e}{}_{;e} + 6 h^{1\,cd} h^{1}{}_{ac;b} h^{2}{}_{d}{}^{e}{}_{;e} - 6 h^{1\,cd} h^{1}{}_{ab;c} h^{2}{}_{d}{}^{e}{}_{;e} +$$
$$6 h^{1\,cd} h^{2}{}_{c}{}^{e} h^{1}{}_{bd;a;e} + 6 h^{1}{}_{c}{}^{e} h^{1\,cd} h^{2}{}_{bd;a;e} + 6 h^{1\,cd} h^{2}{}_{c}{}^{e} h^{1}{}_{ad;b;e} + 6 h^{1}{}_{c}{}^{e} h^{1\,cd} h^{2}{}_{ad;b;e} -$$
$$6 h^{1\,cd} h^{2}{}_{c}{}^{e} h^{1}{}_{ab;d;e} - 6 h^{1}{}_{c}{}^{e} h^{1\,cd} h^{2}{}_{ab;d;e} - 6 h^{2\,cd} h^{1}{}_{ce;d} h^{1}{}_{ab}{}^{;e} - 6 h^{1\,cd} h^{2}{}_{ce;d} h^{1}{}_{ab}{}^{;e} +$$
$$3 h^{2\,cd} h^{1}{}_{cd;e} h^{1}{}_{ab}{}^{;e} + 3 h^{1\,cd} h^{2}{}_{cd;e} h^{1}{}_{ab}{}^{;e} + 6 h^{2\,cd} h^{1}{}_{be;d} h^{1}{}_{ac}{}^{;e} + 6 h^{1\,cd} h^{2}{}_{be;d} h^{1}{}_{ac}{}^{;e} -$$
$$6 h^{2\,cd} h^{1}{}_{bd;e} h^{1}{}_{ac}{}^{;e} - 6 h^{1\,cd} h^{2}{}_{bd;e} h^{1}{}_{ac}{}^{;e} - 6 h^{1\,cd} h^{1}{}_{ce;d} h^{2}{}_{ab}{}^{;e} + 3 h^{1\,cd} h^{1}{}_{cd;e} h^{2}{}_{ab}{}^{;e} +$$
$$6 h^{1\,cd} h^{1}{}_{be;d} h^{2}{}_{ac}{}^{;e} - 6 h^{1\,cd} h^{1}{}_{bd;e} h^{2}{}_{ac}{}^{;e} + 12 h^{1\,cd} h^{1\,ef} h^{1}{}_{ac;e} h^{1}{}_{bd;f} - 12 h^{1}{}_{c}{}^{e} h^{1\,cd} h^{1}{}_{a}{}^{f}{}_{;d} h^{1}{}_{be;f} -$$
$$12 h^{1\,cd} h^{1\,ef} h^{1}{}_{bc;a} h^{1}{}_{de;f} + 6 h^{1}{}_{c}{}^{e} h^{1\,cd} h^{1}{}_{b}{}^{f}{}_{;a} h^{1}{}_{de;f} - 12 h^{1\,cd} h^{1\,ef} h^{1}{}_{ac;b} h^{1}{}_{de;f} +$$
$$6 h^{1}{}_{c}{}^{e} h^{1\,cd} h^{1}{}_{a}{}^{f}{}_{;b} h^{1}{}_{de;f} + 12 h^{1\,cd} h^{1\,ef} h^{1}{}_{ab;c} h^{1}{}_{de;f} - 12 h^{1}{}_{c}{}^{e} h^{1\,cd} h^{1}{}_{bd;a} h^{1}{}_{e}{}^{f}{}_{;f} -$$
$$12 h^{1}{}_{c}{}^{e} h^{1\,cd} h^{1}{}_{ad;b} h^{1}{}_{e}{}^{f}{}_{;f} + 12 h^{1}{}_{c}{}^{e} h^{1\,cd} h^{1}{}_{ab;d} h^{1}{}_{e}{}^{f}{}_{;f} - 12 h^{1}{}_{c}{}^{e} h^{1\,cd} h^{1}{}_{d}{}^{f} h^{1}{}_{be;a;f} -$$
$$12 h^{1}{}_{c}{}^{e} h^{1\,cd} h^{1}{}_{d}{}^{f} h^{1}{}_{ae;b;f} + 12 h^{1}{}_{c}{}^{e} h^{1\,cd} h^{1}{}_{d}{}^{f} h^{1}{}_{ab;e;f} + 12 h^{1}{}_{c}{}^{e} h^{1\,cd} h^{1}{}_{df;e} h^{1}{}_{ab}{}^{;f} -$$
$$6 h^{1}{}_{c}{}^{e} h^{1\,cd} h^{1}{}_{de;f} h^{1}{}_{ab}{}^{;f} - 12 h^{1}{}_{c}{}^{e} h^{1\,cd} h^{1}{}_{bf;e} h^{1}{}_{ad}{}^{;f} + 12 h^{1}{}_{c}{}^{e} h^{1\,cd} h^{1}{}_{be;f} h^{1}{}_{ad}{}^{;f}$$

Some more examples:

In[119]:=
```
Perturbation[RicciCD[-a, -b], 8] // ExpandPerturbation;
```
1.39609 Second

In[120]:=
```
Perturbation[RicciCD[-a, -b], 12] // ExpandPerturbation;
```
38.4104 Second

3.6. Perturbations of the Ricci scalar

The Ricci scalar is computed from a double trace of the Riemann tensor.

In[121]:=

```
ricciscalar =
 g[a, b] RiemannCD[-a, -c, -b, c] // RiemannToChristoffel // ChristoffelToMetric //
  Simplification
```

0.624039 Second

Out[121]=

$$-\frac{1}{4} g^{ab} g^{cd} \left(-4 g_{ac,b,d} + 4 g_{ab,c,d} + g^{ef} \left(2 g_{bf,d} g_{ac,e} - 3 g_{ac,e} g_{bd,f} + g_{ab,c} \left(g_{ef,d} - 4 g_{de,f}\right) + 4 g_{ac,b} g_{de,f}\right)\right)$$

First−order perturbation from the metric

In[122]:=

```
ChangeCovD[Perturbation[ricciscalar] // ExpandPerturbation, PD, CD] // org
```

1.89212 Second

Out[122]=

$$-\Gamma^{abc} \Gamma_{ba}{}^{d} h^{1}{}_{cd} + \Gamma^{a}{}_{a}{}^{b} \Gamma_{b}{}^{cd} h^{1}{}_{cd} + h^{1\,ab} \Gamma^{c}{}_{ac;b} + h^{1\,ab}{}_{;a;b} - h^{1\,a}{}_{a}{}^{;b}{}_{;b} - h^{1\,ab} \Gamma^{c}{}_{ab;c}$$

Applying directly `ExpandPerturbation`

In[123]:=

```
Perturbation[RicciScalarCD[]] // ExpandPerturbation // org
```

Out[123]=

$$-h^{1\,ab} R_{ab} + h^{1\,ab}{}_{;a;b} - h^{1\,a}{}_{a}{}^{;b}{}_{;b}$$

Comparison

In[124]:=

```
ChangeCovD[% - %% // RiemannToChristoffel, PD, CD] // Simplification
```

Out[124]=

0

Second−order perturbation from the metric

In[125]:=

```
ChangeCovD[Perturbation[ricciscalar, 2] // ExpandPerturbation, PD, CD] // org
```

7.17245 Second

Out[125]=

$$2 \Gamma^{abc} \Gamma_{ba}{}^{d} h^{1}{}_{c}{}^{e} h^{1}{}_{de} - 2 \Gamma^{a}{}_{a}{}^{b} \Gamma_{b}{}^{cd} h^{1}{}_{c}{}^{e} h^{1}{}_{de} - \Gamma^{abc} \Gamma_{ba}{}^{d} h^{2}{}_{cd} + \Gamma^{a}{}_{a}{}^{b} \Gamma_{b}{}^{cd} h^{2}{}_{cd} + h^{2\,ab} \Gamma^{c}{}_{ac;b} + 2 h^{1\,ab} h^{1\,c}{}_{c;a;b} + h^{2\,ab}{}_{;a;b} - h^{2\,a}{}_{a}{}^{;b}{}_{;b} - 2 h^{1\,ab} h^{1}{}_{a}{}^{c}{}_{;c;b} - \frac{1}{2} h^{1\,c}{}_{c;b} h^{1\,a}{}_{a}{}^{;b} - h^{2\,ab} \Gamma^{c}{}_{ab;c} - 2 h^{1}{}_{a}{}^{c} h^{1\,ab} \Gamma^{d}{}_{bd;c} - 2 h^{1\,ab}{}_{;a} h^{1}{}_{b}{}^{c}{}_{;c} + 2 h^{1\,a}{}_{a}{}^{;b} h^{1}{}_{b}{}^{c}{}_{;c} - 2 h^{1\,ab} h^{1}{}_{a}{}^{c}{}_{;b;c} + 2 h^{1\,ab} h^{1}{}_{ab}{}^{;c}{}_{;c} - h^{1}{}_{ac;b} h^{1\,ab;c} + \frac{3}{2} h^{1}{}_{ab;c} h^{1\,ab;c} + 2 h^{1}{}_{a}{}^{c} h^{1\,ab} \Gamma^{d}{}_{bc;d}$$

Applying directly `ExpandPerturbation`

In[126]:=

```
Perturbation[RicciScalarCD[], 2] // ExpandPerturbation // org
```

0.368023 Second

Out[126]=

$$-h^{2\,ab} R_{ab} + 2 h^{1}{}_{a}{}^{c} h^{1\,ab} R_{bc} + 2 h^{1\,ab} h^{1\,c}{}_{c;a;b} + h^{2\,ab}{}_{;a;b} - h^{2\,a}{}_{a}{}^{;b}{}_{;b} - 2 h^{1\,ab} h^{1}{}_{a}{}^{c}{}_{;c;b} - \frac{1}{2} h^{1\,c}{}_{c;b} h^{1\,a}{}_{a}{}^{;b} - 2 h^{1\,ab}{}_{;a} h^{1}{}_{b}{}^{c}{}_{;c} + 2 h^{1\,a}{}_{a}{}^{;b} h^{1}{}_{b}{}^{c}{}_{;c} - 2 h^{1\,ab} h^{1}{}_{a}{}^{c}{}_{;b;c} + 2 h^{1\,ab} h^{1}{}_{ab}{}^{;c}{}_{;c} - h^{1}{}_{ac;b} h^{1\,ab;c} + \frac{3}{2} h^{1}{}_{ab;c} h^{1\,ab;c}$$

Comparison

In[127]:=

```
ChangeCovD[% - %% // RiemannToChristoffel, PD, CD] // Simplification
```

0.208013 Second

Out[127]=

0

Third−order perturbation iteratively from metric perturbations

In[128]:=

```
ChangeCovD[Perturbation[ricciscalar, 3] // ExpandPerturbation, PD, CD] // org
```

27.6217 Second

Out[128]=

$$-6 \Gamma^{abc} \Gamma_{ba}{}^{d} h^{1}{}_{c}{}^{e} h^{1}{}_{d}{}^{f} h^{1}{}_{ef} + 6 \Gamma^{a}{}_{a}{}^{b} \Gamma_{b}{}^{cd} h^{1}{}_{c}{}^{e} h^{1}{}_{d}{}^{f} h^{1}{}_{ef} + 6 \Gamma^{abc} \Gamma_{ba}{}^{d} h^{1}{}_{c}{}^{e} h^{2}{}_{de} - 6 \Gamma^{a}{}_{a}{}^{b} \Gamma_{b}{}^{cd} h^{1}{}_{c}{}^{e} h^{2}{}_{de} -$$
$$\Gamma^{abc} \Gamma_{ba}{}^{d} h^{3}{}_{cd} + \Gamma^{a}{}_{a}{}^{b} \Gamma_{b}{}^{cd} h^{3}{}_{cd} + h^{3\,ab} \Gamma^{c}{}_{ac;b} - 3 h^{1\,ab} h^{2}{}_{a}{}^{c} \Gamma^{d}{}_{cd;b} - \frac{9}{2} h^{1\,ab} h^{1\,cd}{}_{;a} h^{1}{}_{cd;b} +$$
$$\frac{3}{2} h^{1\,ab} h^{1\,c}{}_{c;a} h^{1\,d}{}_{d;b} + 3 h^{1\,ab}{}_{;a} h^{2\,c}{}_{c;b} + 3 h^{2\,ab} h^{1\,c}{}_{c;a;b} + 3 h^{1\,ab} h^{2\,c}{}_{c;a;b} + h^{3\,ab}{}_{;a;b} - h^{3\,a}{}_{a}{}^{;b}{}_{;b} -$$
$$3 h^{2\,ab} h^{1}{}_{a}{}^{c}{}_{;c;b} - 3 h^{1\,ab} h^{2}{}_{a}{}^{c}{}_{;c;b} - \frac{3}{2} h^{2\,c}{}_{c;b} h^{1\,a}{}_{a}{}^{;b} - h^{3\,ab} \Gamma^{c}{}_{ab;c} - 3 h^{1\,ab} h^{2}{}_{a}{}^{c} \Gamma^{d}{}_{bd;c} -$$
$$6 h^{1\,ab} h^{1\,d}{}_{d;b} h^{1}{}_{a}{}^{c}{}_{;c} - 6 h^{1\,ab} h^{1}{}_{a}{}^{c}{}_{;b} h^{1\,d}{}_{d;c} - 6 h^{1\,ab}{}_{;a} h^{2}{}_{b}{}^{c}{}_{;c} + 3 h^{1\,a}{}_{a}{}^{;b} h^{2}{}_{b}{}^{c}{}_{;c} - 3 h^{2\,ab} h^{1}{}_{a}{}^{c}{}_{;b;c} -$$
$$6 h^{1}{}_{a}{}^{c} h^{1\,ab} h^{1\,d}{}_{d;b;c} - 3 h^{1\,ab} h^{2}{}_{a}{}^{c}{}_{;b;c} + 3 h^{2\,ab} h^{1}{}_{ab}{}^{;c}{}_{;c} + 3 h^{1\,ab} h^{2}{}_{ab}{}^{;c}{}_{;c} + 6 h^{1}{}_{a}{}^{c} h^{1\,ab} h^{1}{}_{b}{}^{d}{}_{;d;c} +$$
$$3 h^{1\,ab} h^{1\,d}{}_{d;c} h^{1}{}_{ab}{}^{;c} - 3 h^{2}{}_{ac;b} h^{1\,ab;c} + \frac{9}{2} h^{2}{}_{ab;c} h^{1\,ab;c} + 6 h^{1\,ab} h^{2}{}_{a}{}^{c} \Gamma^{d}{}_{bc;d} +$$
$$6 h^{1}{}_{a}{}^{c} h^{1\,ab} h^{1}{}_{b}{}^{d} \Gamma^{e}{}_{ce;d} + 6 h^{1\,ab} h^{1}{}_{a}{}^{c}{}_{;c} h^{1}{}_{b}{}^{d}{}_{;d} + 12 h^{1\,ab} h^{1}{}_{a}{}^{c}{}_{;b} h^{1}{}_{c}{}^{d}{}_{;d} - 6 h^{1\,ab} h^{1}{}_{ab}{}^{;c} h^{1}{}_{c}{}^{d}{}_{;d} +$$
$$6 h^{1\,ab} h^{1\,cd} h^{1}{}_{ac;b;d} - 6 h^{1\,ab} h^{1\,cd} h^{1}{}_{ab;c;d} + 6 h^{1}{}_{a}{}^{c} h^{1\,ab} h^{1}{}_{b}{}^{d}{}_{;c;d} - 6 h^{1}{}_{a}{}^{c} h^{1\,ab} h^{1}{}_{bc}{}^{;d}{}_{;d} +$$
$$6 h^{1\,ab} h^{1}{}_{cd;b} h^{1}{}_{a}{}^{c;d} + 3 h^{1\,ab} h^{1}{}_{bd;c} h^{1}{}_{a}{}^{c;d} - 9 h^{1\,ab} h^{1}{}_{bc;d} h^{1}{}_{a}{}^{c;d} - 6 h^{1}{}_{a}{}^{c} h^{1\,ab} h^{1}{}_{b}{}^{d} \Gamma^{e}{}_{cd;e}$$

Applying directly `ExpandPerturbation`

In[129]:=
```
Perturbation[RicciScalarCD[], 3] // ExpandPerturbation // org
```

`1.5401 Second`

Out[129]=

$$\begin{aligned}
&-h^{3\,ab}\,R_{ab}+6\,h^{1\,ab}\,h^{2}{}_{a}{}^{c}\,R_{bc}-6\,h^{1}{}_{a}{}^{c}\,h^{1\,ab}\,h^{1}{}_{b}{}^{d}\,R_{cd}-\frac{9}{2}\,h^{1\,ab}\,h^{1\,cd}{}_{;a}\,h^{1}{}_{cd;b}+\\
&\frac{3}{2}\,h^{1\,ab}\,h^{1\,c}{}_{c;a}\,h^{1\,d}{}_{d;b}+3\,h^{1\,ab}{}_{;a}\,h^{2\,c}{}_{c;b}+3\,h^{2\,ab}\,h^{1\,c}{}_{c;a;b}+3\,h^{1\,ab}\,h^{2\,c}{}_{c;a;b}+h^{3\,ab}{}_{;a;b}-\\
&h^{3\,a}{}_{a}{}^{;b}{}_{;b}-3\,h^{2\,ab}\,h^{1}{}_{a}{}^{c}{}_{;c;b}-3\,h^{1\,ab}\,h^{2}{}_{a}{}^{c}{}_{;c;b}-\frac{3}{2}\,h^{2\,c}{}_{c;b}\,h^{1\,a}{}_{a}{}^{;b}-6\,h^{1\,ab}\,h^{1\,d}{}_{d;b}\,h^{1}{}_{a}{}^{c}{}_{;c}-\\
&6\,h^{1\,ab}\,h^{1}{}_{a}{}^{c}{}_{;b}\,h^{1\,d}{}_{d;c}-6\,h^{1\,ab}{}_{;a}\,h^{2}{}_{b}{}^{c}{}_{;c}+3\,h^{1\,a}{}_{a}{}^{;b}\,h^{2}{}_{b}{}^{c}{}_{;c}-3\,h^{2\,ab}\,h^{1}{}_{a}{}^{c}{}_{;b;c}-6\,h^{1}{}_{a}{}^{c}\,h^{1\,ab}\,h^{1\,d}{}_{d;b;c}-\\
&3\,h^{1\,ab}\,h^{2}{}_{a}{}^{c}{}_{;b;c}+3\,h^{2\,ab}\,h^{1}{}_{ab}{}^{;c}{}_{;c}+3\,h^{1\,ab}\,h^{2}{}_{ab}{}^{;c}{}_{;c}+6\,h^{1}{}_{a}{}^{c}\,h^{1\,ab}\,h^{1}{}_{b}{}^{d}{}_{;d;c}+3\,h^{1\,ab}\,h^{1\,d}{}_{d;c}\,h^{1}{}_{ab}{}^{;c}-\\
&3\,h^{2}{}_{ac;b}\,h^{1\,ab;c}+\frac{9}{2}\,h^{2}{}_{ab;c}\,h^{1\,ab;c}+6\,h^{1\,ab}\,h^{1}{}_{a}{}^{c}{}_{;c}\,h^{1}{}_{b}{}^{d}{}_{;d}+12\,h^{1\,ab}\,h^{1}{}_{a}{}^{c}{}_{;b}\,h^{1}{}_{c}{}^{d}{}_{;d}-\\
&6\,h^{1\,ab}\,h^{1}{}_{ab}{}^{;c}\,h^{1}{}_{c}{}^{d}{}_{;d}+6\,h^{1\,ab}\,h^{1\,cd}\,h^{1}{}_{ac;b;d}-6\,h^{1\,ab}\,h^{1\,cd}\,h^{1}{}_{ab;c;d}+6\,h^{1}{}_{a}{}^{c}\,h^{1\,ab}\,h^{1}{}_{b}{}^{d}{}_{;c;d}-\\
&6\,h^{1}{}_{a}{}^{c}\,h^{1\,ab}\,h^{1}{}_{bc}{}^{;d}{}_{;d}+6\,h^{1\,ab}\,h^{1}{}_{cd;b}\,h^{1}{}_{a}{}^{c;d}+3\,h^{1\,ab}\,h^{1}{}_{bd;c}\,h^{1}{}_{a}{}^{c;d}-9\,h^{1\,ab}\,h^{1}{}_{bc;d}\,h^{1}{}_{a}{}^{c;d}
\end{aligned}$$

Comparison

In[130]:=
```
ChangeCovD[%% - % // RiemannToChristoffel, PD, CD] // org
```

`0.456029 Second`

Out[130]=

$$3\,h^{1\,ab}\,h^{2}{}_{a}{}^{c}\,\Gamma^{d}{}_{cd;b}-3\,h^{1\,ab}\,h^{2}{}_{a}{}^{c}\,\Gamma^{d}{}_{bd;c}$$

That expression is indeed only zero for a Levi−Civita connection. To show it we change back to partial derivatives and expand the Christoffels into derivatives of the metric:

In[131]:=
```
ChangeCovD[%, CD, PD] // ToCanonical
```

Out[131]=

$$3\,h^{1\,ab}\,h^{2}{}_{a}{}^{c}\,\Gamma^{d}{}_{cd,b}-3\,h^{1\,ab}\,h^{2}{}_{a}{}^{c}\,\Gamma^{d}{}_{bd,c}$$

In[132]:=
```
% // ChristoffelToMetric // ToCanonical
```

`0.228014 Second`

Out[132]=

`0`

Fourth−order perturbation iteratively from metric perturbations

In[133]:=
```
ChangeCovD[Perturbation[ricciscalar, 4] // ExpandPerturbation, PD, CD] // org;
```

`83.7492 Second`

Applying directly `ExpandPerturbation`

In[134]:=
```
Perturbation[RicciScalarCD[], 4] // ExpandPerturbation // org;
```

5.08832 Second

Comparison

In[135]:=
```
ChangeCovD[%% - % // ToCanonical // RiemannToChristoffel, PD, CD] // Simplification
```

1.38009 Second

Out[135]=
$$4\, h^{1\,ab}\, (h^{3}{}_{a}{}^{c}\, (\Gamma^{d}{}_{cd;b} - \Gamma^{d}{}_{bd;c}) + 3\, h^{1}{}_{a}{}^{c}\, h^{2}{}_{b}{}^{d}\, (-\Gamma^{e}{}_{de;c} + \Gamma^{e}{}_{ce;d}))$$

Again, this is zero for a Levi−Civita connection (we do not use the metric in the index−canonicalization process to avoid problems with the partial derivatives):

In[136]:=
```
Simplification[ChangeCovD[%, CD, PD], UseMetricOnVBundle → None]
```

0.108007 Second

Out[136]=
$$4\, h^{1\,ab}\, (h^{3}{}_{a}{}^{c}\, (\Gamma^{d}{}_{cd,b} - \Gamma^{d}{}_{bd,c}) + 3\, h^{1}{}_{a}{}^{c}\, h^{2}{}_{b}{}^{d}\, (-\Gamma^{e}{}_{de,c} + \Gamma^{e}{}_{ce,d}))$$

In[137]:=
```
Simplification[% // ChristoffelToMetric, UseMetricOnVBundle → None]
```

0.440028 Second

Out[137]=
0

3.7. Perturbations of the Einstein tensor

To help in the recursive computation, we define the Einstein tensor in terms of the Christoffels instead of the metric.

In[138]:=
```
einstein = EinsteinCD[-a, -b] // EinsteinToRicci // RiemannToChristoffel // Simplify
```

Out[138]=
$$\Gamma^{c}{}_{cd}\, \Gamma^{d}{}_{ab} - \Gamma^{c}{}_{ad}\, \Gamma^{d}{}_{cb} - \Gamma^{c}{}_{cb,a} + \Gamma^{c}{}_{ab,c} - \frac{1}{2}\, g_{ba}\, g^{cd}\, (\Gamma^{e}{}_{ef}\, \Gamma^{f}{}_{cd} - \Gamma^{e}{}_{cf}\, \Gamma^{f}{}_{ed} - \Gamma^{e}{}_{ed,c} + \Gamma^{e}{}_{cd,e})$$

The first−order perturbation from the Christoffel perturbations:

In[139]:=
```
Perturbation[einstein] // ExpandPerturbation
```

Out[139]=

$$-\frac{1}{2}\Gamma^{d}{}_{cb}\,(h^{1\,c}{}_{d;a}-h^{1}{}_{ad}{}^{;c}+h^{1\,c}{}_{a;d})+\frac{1}{2}\Gamma^{d}{}_{ab}\,(h^{1\,c}{}_{d;c}-h^{1}{}_{cd}{}^{;c}+h^{1\,c}{}_{c;d})+\frac{1}{2}\Gamma^{c}{}_{cd}\,(h^{1\,d}{}_{b;a}+h^{1\,d}{}_{a;b}-h^{1}{}_{ab}{}^{;d})-\frac{1}{2}\Gamma^{c}{}_{ad}\,(h^{1\,d}{}_{c;b}+h^{1\,d}{}_{b;c}-h^{1}{}_{cb}{}^{;d})+\frac{1}{2}(-h^{1\,c}{}_{c;b,a}-h^{1\,c}{}_{b;c,a}+h^{1}{}_{cb}{}^{;c}{}_{,a})+\frac{1}{2}(h^{1\,c}{}_{b;a,c}+h^{1\,c}{}_{a;b,c}-h^{1}{}_{ab}{}^{;c}{}_{,c})+\frac{1}{2}\Big(-g^{cd}\,h^{1}{}_{ba}\,(\Gamma^{e}{}_{ef}\,\Gamma^{f}{}_{cd}-\Gamma^{e}{}_{cf}\,\Gamma^{f}{}_{ed}-\Gamma^{e}{}_{ed,c}+\Gamma^{e}{}_{cd,e})+g_{ba}\,h^{1\,cd}\,(\Gamma^{e}{}_{ef}\,\Gamma^{f}{}_{cd}-\Gamma^{e}{}_{cf}\,\Gamma^{f}{}_{ed}-\Gamma^{e}{}_{ed,c}+\Gamma^{e}{}_{cd,e})-g_{ba}\,g^{cd}\Big(-\frac{1}{2}\Gamma^{f}{}_{ed}\,(h^{1\,e}{}_{f;c}-h^{1}{}_{cf}{}^{;e}+h^{1\,e}{}_{c;f})+\frac{1}{2}\Gamma^{f}{}_{cd}\,(h^{1\,e}{}_{f;e}-h^{1}{}_{ef}{}^{;e}+h^{1\,e}{}_{e;f})+\frac{1}{2}\Gamma^{e}{}_{ef}\,(h^{1\,f}{}_{d;c}+h^{1\,f}{}_{c;d}-h^{1}{}_{cd}{}^{;f})-\frac{1}{2}\Gamma^{e}{}_{cf}\,(h^{1\,f}{}_{e;d}+h^{1\,f}{}_{d;e}-h^{1}{}_{ed}{}^{;f})+\frac{1}{2}(-h^{1\,e}{}_{e;d,c}-h^{1\,e}{}_{d;e,c}+h^{1}{}_{ed}{}^{;e}{}_{,c})+\frac{1}{2}(h^{1\,e}{}_{d;c,e}+h^{1\,e}{}_{c;d,e}-h^{1}{}_{cd}{}^{;e}{}_{,e})\Big)\Big)$$

Appliying directly `ExpandPerturbation`

In[140]:=
```
Perturbation[EinsteinCD[-a, -b]] // ExpandPerturbation
```

Out[140]=

$$\frac{1}{2}(-h^{1\,c}{}_{c;b;a}-h^{1\,c}{}_{b;c;a}+h^{1}{}_{bc}{}^{;c}{}_{;a})+\frac{1}{2}(h^{1\,c}{}_{b;a;c}+h^{1\,c}{}_{a;b;c}-h^{1}{}_{ba}{}^{;c}{}_{;c})+\frac{1}{2}\Big(-h^{1}{}_{ab}\,R-g_{ab}\Big(-h^{1\,cd}\,R_{cd}+g^{cd}\Big(\frac{1}{2}(-h^{1\,e}{}_{e;d;c}-h^{1\,e}{}_{d;e;c}+h^{1}{}_{de}{}^{;e}{}_{;c})+\frac{1}{2}(h^{1\,e}{}_{d;c;e}+h^{1\,e}{}_{c;d;e}-h^{1}{}_{dc}{}^{;e}{}_{;e})\Big)\Big)\Big)$$

Comparison

In[141]:=
```
ChangeCovD[%% - % // RiemannToChristoffel, PD, CD] // ToCanonical
```

0.580037 Second

Out[141]=

0

The second−order perturbation from Christoffel perturbations:

In[142]:=

```
Perturbation[einstein, 2] // ExpandPerturbation
```

Out[142]=

$$
\begin{aligned}
&\frac{1}{2}\left(h^{1\,c}{}_{d;c}-h^{1}{}_{cd}{}^{;c}+h^{1\,c}{}_{c;d}\right)\left(h^{1\,d}{}_{b;a}+h^{1\,d}{}_{a;b}-h^{1}{}_{ab}{}^{;d}\right)-\\
&\frac{1}{2}\left(h^{1\,c}{}_{d;a}-h^{1}{}_{ad}{}^{;c}+h^{1\,c}{}_{a;d}\right)\left(h^{1\,d}{}_{c;b}+h^{1\,d}{}_{b;c}-h^{1}{}_{cb}{}^{;d}\right)+\\
&\Gamma^{c}{}_{cd}\left(\frac{1}{2}\left(h^{2\,d}{}_{b;a}+h^{2\,d}{}_{a;b}-h^{2}{}_{ab}{}^{;d}\right)-h^{1\,de}\left(h^{1}{}_{eb;a}+h^{1}{}_{ea;b}-h^{1}{}_{ab;e}\right)\right)-\\
&\Gamma^{d}{}_{cb}\left(\frac{1}{2}\left(h^{2\,c}{}_{d;a}-h^{2}{}_{ad}{}^{;c}+h^{2\,c}{}_{a;d}\right)-h^{1\,ce}\left(h^{1}{}_{ed;a}+h^{1}{}_{ea;d}-h^{1}{}_{ad;e}\right)\right)-\\
&\Gamma^{c}{}_{ad}\left(\frac{1}{2}\left(h^{2\,d}{}_{c;b}+h^{2\,d}{}_{b;c}-h^{2}{}_{cb}{}^{;d}\right)-h^{1\,de}\left(h^{1}{}_{ec;b}+h^{1}{}_{eb;c}-h^{1}{}_{cb;e}\right)\right)+\\
&\Gamma^{d}{}_{ab}\left(\frac{1}{2}\left(h^{2\,c}{}_{d;c}-h^{2}{}_{cd}{}^{;c}+h^{2\,c}{}_{c;d}\right)-h^{1\,ce}\left(h^{1}{}_{ed;c}+h^{1}{}_{ec;d}-h^{1}{}_{cd;e}\right)\right)+\\
&\left(h^{1}{}_{dc;b}+h^{1}{}_{db;c}-h^{1}{}_{cb;d}\right)h^{1\,cd}{}_{,a}+\frac{1}{2}\left(-h^{2\,c}{}_{c;b,a}-h^{2\,c}{}_{b;c,a}+h^{2}{}_{cb}{}^{;c}{}_{,a}\right)+\\
&h^{1\,cd}\left(h^{1}{}_{dc;b,a}+h^{1}{}_{db;c,a}-h^{1}{}_{cb;d,a}\right)-\left(h^{1}{}_{db;a}+h^{1}{}_{da;b}-h^{1}{}_{ab;d}\right)h^{1\,cd}{}_{,c}+\\
&\frac{1}{2}\left(h^{2\,c}{}_{b;a,c}+h^{2\,c}{}_{a;b,c}-h^{2}{}_{ab}{}^{;c}{}_{,c}\right)-h^{1\,cd}\left(h^{1}{}_{db;a,c}+h^{1}{}_{da;b,c}-h^{1}{}_{ab;d,c}\right)+\frac{1}{2}\\
&\quad\Big(2\,h^{1}{}_{ba}\,h^{1\,cd}\left(\Gamma^{e}{}_{ef}\Gamma^{f}{}_{cd}-\Gamma^{e}{}_{cf}\Gamma^{f}{}_{ed}-\Gamma^{e}{}_{ed,c}+\Gamma^{e}{}_{cd,e}\right)-g^{cd}\,h^{2}{}_{ba}\left(\Gamma^{e}{}_{ef}\Gamma^{f}{}_{cd}-\Gamma^{e}{}_{cf}\Gamma^{f}{}_{ed}-\Gamma^{e}{}_{ed,c}+\Gamma^{e}{}_{cd,e}\right)-\\
&\quad g_{ba}\left(2\,h^{1\,cf1}\,h^{1}{}_{f1}{}^{d}-h^{2\,cd}\right)\left(\Gamma^{e}{}_{ef}\Gamma^{f}{}_{cd}-\Gamma^{e}{}_{cf}\Gamma^{f}{}_{ed}-\Gamma^{e}{}_{ed,c}+\Gamma^{e}{}_{cd,e}\right)-\\
&\quad 2\,g^{cd}\,h^{1}{}_{ba}\Big(-\frac{1}{2}\Gamma^{f}{}_{ed}\left(h^{1\,e}{}_{f;c}-h^{1}{}_{cf}{}^{;e}+h^{1\,e}{}_{c;f}\right)+\frac{1}{2}\Gamma^{f}{}_{cd}\left(h^{1\,e}{}_{f;e}-h^{1}{}_{ef}{}^{;e}+h^{1\,e}{}_{e;f}\right)+\\
&\qquad\frac{1}{2}\Gamma^{e}{}_{ef}\left(h^{1\,f}{}_{d;c}+h^{1\,f}{}_{c;d}-h^{1}{}_{cd}{}^{;f}\right)-\frac{1}{2}\Gamma^{e}{}_{cf}\left(h^{1\,f}{}_{e;d}+h^{1\,f}{}_{d;e}-h^{1}{}_{ed}{}^{;f}\right)+\\
&\qquad\frac{1}{2}\left(-h^{1\,e}{}_{e;d,c}-h^{1\,e}{}_{d;e,c}+h^{1}{}_{ed}{}^{;e}{}_{,c}\right)+\frac{1}{2}\left(h^{1\,e}{}_{d;c,e}+h^{1\,e}{}_{c;d,e}-h^{1}{}_{cd}{}^{;e}{}_{,e}\right)\Big)+\\
&\quad 2\,g_{ba}\,h^{1\,cd}\Big(-\frac{1}{2}\Gamma^{f}{}_{ed}\left(h^{1\,e}{}_{f;c}-h^{1}{}_{cf}{}^{;e}+h^{1\,e}{}_{c;f}\right)+\frac{1}{2}\Gamma^{f}{}_{cd}\left(h^{1\,e}{}_{f;e}-h^{1}{}_{ef}{}^{;e}+h^{1\,e}{}_{e;f}\right)+\\
&\qquad\frac{1}{2}\Gamma^{e}{}_{ef}\left(h^{1\,f}{}_{d;c}+h^{1\,f}{}_{c;d}-h^{1}{}_{cd}{}^{;f}\right)-\frac{1}{2}\Gamma^{e}{}_{cf}\left(h^{1\,f}{}_{e;d}+h^{1\,f}{}_{d;e}-h^{1}{}_{ed}{}^{;f}\right)+\\
&\qquad\frac{1}{2}\left(-h^{1\,e}{}_{e;d,c}-h^{1\,e}{}_{d;e,c}+h^{1}{}_{ed}{}^{;e}{}_{,c}\right)+\frac{1}{2}\left(h^{1\,e}{}_{d;c,e}+h^{1\,e}{}_{c;d,e}-h^{1}{}_{cd}{}^{;e}{}_{,e}\right)\Big)-g_{ba}\,g^{cd}\\
&\quad\Big(\frac{1}{2}\left(h^{1\,e}{}_{f;e}-h^{1}{}_{ef}{}^{;e}+h^{1\,e}{}_{e;f}\right)\left(h^{1\,f}{}_{d;c}+h^{1\,f}{}_{c;d}-h^{1}{}_{cd}{}^{;f}\right)-\frac{1}{2}\left(h^{1\,e}{}_{f;c}-h^{1}{}_{cf}{}^{;e}+h^{1\,e}{}_{c;f}\right)\Big(h^{1\,f}{}_{e;d}+\\
&\qquad\quad h^{1\,f}{}_{d;e}-h^{1}{}_{ed}{}^{;f}\Big)-\Gamma^{f}{}_{ed}\left(\frac{1}{2}\left(h^{2\,e}{}_{f;c}-h^{2}{}_{cf}{}^{;e}+h^{2\,e}{}_{c;f}\right)-h^{1\,ef10}\left(h^{1}{}_{f10f;c}+h^{1}{}_{f10c;f}-h^{1}{}_{cf;f10}\right)\right)\\
&\qquad\Gamma^{f}{}_{cd}\left(\frac{1}{2}\left(h^{2\,e}{}_{f;e}-h^{2}{}_{ef}{}^{;e}+h^{2\,e}{}_{e;f}\right)-h^{1\,ef11}\left(h^{1}{}_{f11f;e}+h^{1}{}_{f11e;f}-h^{1}{}_{ef;f11}\right)\right)+\\
&\qquad\Gamma^{e}{}_{ef}\left(\frac{1}{2}\left(h^{2\,f}{}_{d;c}+h^{2\,f}{}_{c;d}-h^{2}{}_{cd}{}^{;f}\right)-h^{1\,ff12}\left(h^{1}{}_{f12d;c}+h^{1}{}_{f12c;d}-h^{1}{}_{cd;f12}\right)\right)-\\
&\qquad\Gamma^{e}{}_{cf}\left(\frac{1}{2}\left(h^{2\,f}{}_{e;d}+h^{2\,f}{}_{d;e}-h^{2}{}_{ed}{}^{;f}\right)-h^{1\,ff13}\left(h^{1}{}_{f13e;d}+h^{1}{}_{f13d;e}-h^{1}{}_{ed;f13}\right)\right)+\\
&\qquad\left(h^{1}{}_{f14e;d}+h^{1}{}_{f14d;e}-h^{1}{}_{ed;f14}\right)h^{1\,ef14}{}_{,c}+\frac{1}{2}\left(-h^{2\,e}{}_{e;d,c}-h^{2\,e}{}_{d;e,c}+h^{2}{}_{ed}{}^{;e}{}_{,c}\right)+\\
&\qquad h^{1\,ef14}\left(h^{1}{}_{f14e;d,c}+h^{1}{}_{f14d;e,c}-h^{1}{}_{ed;f14,c}\right)-\left(h^{1}{}_{f15d;c}+h^{1}{}_{f15c;d}-h^{1}{}_{cd;f15}\right)h^{1\,ef15}{}_{,e}+\\
&\qquad\frac{1}{2}\left(h^{2\,e}{}_{d;c,e}+h^{2\,e}{}_{c;d,e}-h^{2}{}_{cd}{}^{;e}{}_{,e}\right)-h^{1\,ef15}\left(h^{1}{}_{f15d;c,e}+h^{1}{}_{f15c;d,e}-h^{1}{}_{cd;f15,e}\right)\Big)\Big)
\end{aligned}
$$

Applying directly `ExpandPerturbation`

In[143]:=
```
Perturbation[EinsteinCD[-a, -b], 2] // ExpandPerturbation
```

Out[143]=

$$\begin{aligned}
&\frac{1}{2}\,(-h^{2\,c}{}_{c;b;a}-h^{2\,c}{}_{b;c;a}+h^{2}{}_{bc}{}^{;c}{}_{;a})+\frac{1}{2}\,(h^{2\,c}{}_{b;a;c}+h^{2\,c}{}_{a;b;c}-h^{2}{}_{ba}{}^{;c}{}_{;c})+\\
&2\,\Big(\frac{1}{2}\,h^{1\,cd}\,(h^{1}{}_{dc;b;a}+h^{1}{}_{db;c;a}-h^{1}{}_{bc;d;a})+\frac{1}{4}\,(h^{1}{}_{ec;b}+h^{1}{}_{eb;c}-h^{1}{}_{cb;e})\,(h^{1\,ec}{}_{;a}+h^{1\,e}{}_{a}{}^{;c}-h^{1\,c}{}_{a}{}^{;e})\Big)-\\
&2\,\Big(\frac{1}{2}\,h^{1\,cd}\,(h^{1}{}_{db;a;c}+h^{1}{}_{da;b;c}-h^{1}{}_{ba;d;c})+\frac{1}{4}\,(h^{1}{}_{eb;a}+h^{1}{}_{ea;b}-h^{1}{}_{ab;e})\,(h^{1\,ec}{}_{;c}+h^{1\,e}{}_{c}{}^{;c}-h^{1\,c}{}_{c}{}^{;e})\Big)+\\
&\frac{1}{2}\,\Big(-h^{2}{}_{ab}\,R-g_{ab}\,\Big((2\,h^{1\,ce}\,h^{1}{}_{e}{}^{d}-h^{2\,cd})\,R_{cd}-\\
&\quad 2\,h^{1\,cd}\,\Big(\frac{1}{2}\,(-h^{1\,f}{}_{f;d;c}-h^{1\,f}{}_{d;f;c}+h^{1}{}_{df}{}^{;f}{}_{;c})+\frac{1}{2}\,(h^{1\,f}{}_{d;c;f}+h^{1\,f}{}_{c;d;f}-h^{1}{}_{dc}{}^{;f}{}_{;f})\Big)+\\
&\quad g^{cd}\,\Big(\frac{1}{2}\,(-h^{2\,f1}{}_{f1;d;c}-h^{2\,f1}{}_{d;f1;c}+h^{2}{}_{df1}{}^{;f1}{}_{;c})+\frac{1}{2}\,(h^{2\,f1}{}_{d;c;f1}+h^{2\,f1}{}_{c;d;f1}-h^{2}{}_{dc}{}^{;f1}{}_{;f1})+\\
&\qquad 2\,\Big(\frac{1}{2}\,h^{1\,f1f10}\,(h^{1}{}_{f10f1;d;c}+h^{1}{}_{f10d;f1;c}-h^{1}{}_{df1;f10;c})+\\
&\qquad\quad \frac{1}{4}\,(h^{1}{}_{f11f1;d}+h^{1}{}_{f11d;f1}-h^{1}{}_{f1d;f11})\,(h^{1\,f11f1}{}_{;c}+h^{1\,f11}{}_{c}{}^{;f1}-h^{1\,f1}{}_{c}{}^{;f11})\Big)-\\
&\qquad 2\,\Big(\frac{1}{2}\,h^{1\,f1f10}\,(h^{1}{}_{f10d;c;f1}+h^{1}{}_{f10c;d;f1}-h^{1}{}_{dc;f10;f1})+\\
&\qquad\quad \frac{1}{4}\,(h^{1}{}_{f11d;c}+h^{1}{}_{f11c;d}-h^{1}{}_{cd;f11})\,(h^{1\,f11f1}{}_{;f1}+h^{1\,f11}{}_{f1}{}^{;f1}-h^{1\,f1}{}_{f1}{}^{;f11})\Big)\Big)\Big)\Big)-\\
&\quad 2\,h^{1}{}_{ab}\,\Big(-h^{1\,f12f13}\,R_{f12f13}+g^{f12f13}\,\Big(\frac{1}{2}\,(-h^{1\,f14}{}_{f14;f13;f12}-h^{1\,f14}{}_{f13;f14;f12}+h^{1}{}_{f13f14}{}^{;f14}{}_{;f12})+\\
&\qquad \frac{1}{2}\,(h^{1\,f14}{}_{f13;f12;f14}+h^{1\,f14}{}_{f12;f13;f14}-h^{1}{}_{f13f12}{}^{;f14}{}_{;f14})\Big)\Big)\Big)
\end{aligned}$$

Comparison

In[144]:=
```
ChangeCovD[%% - % // RiemannToChristoffel, PD, CD] // ToCanonical
```

```
3.45622 Second
```

Out[144]=
0

Note that `ExpandPerturbation` is really fast. The seventh−order perturbation of the Einstein tensor is computed in just a few seconds.

In[145]:=
```
Perturbation[EinsteinCD[-a, -b], 7] // ExpandPerturbation;
```

```
2.08413 Second
```

3.8. Perturbations of the Weyl tensor

The Weyl tensor is given here in terms of Riemann, Ricci and the Ricci scalar. Giving it in terms of the metric would be too slow.

Weyl in terms of Riemann:

In[146]:=

```
WeylCD[-a, -b, -c, -d] // WeylToRiemann
```

Out[146]=

$$-\frac{1}{2} g_{db} R_{ac} + \frac{1}{2} g_{cb} R_{ad} + \frac{1}{2} g_{da} R_{bc} - \frac{1}{2} g_{ca} R_{bd} - \frac{1}{6} g_{cb} g_{da} R + \frac{1}{6} g_{ca} g_{db} R + R_{abcd}$$

In[147]:=

```
Perturbation[%] // ExpandPerturbation
```

Out[147]=

$$\begin{aligned}
& h^{1}{}_{ed} R_{abc}{}^{e} + g_{ed} \left(\frac{1}{2} \left(-h^{1\,e}{}_{c;b;a} - h^{1\,e}{}_{b;c;a} + h^{1}{}_{cb}{}^{;e}{}_{;a}\right) + \frac{1}{2} \left(h^{1\,e}{}_{c;a;b} + h^{1\,e}{}_{a;c;b} - h^{1}{}_{ca}{}^{;e}{}_{;b}\right)\right) + \\
& \frac{1}{2} \left(-h^{1}{}_{db} R_{ac} - g_{db} \left(\frac{1}{2} \left(-h^{1\,e}{}_{e;c;a} - h^{1\,e}{}_{c;e;a} + h^{1}{}_{ce}{}^{;e}{}_{;a}\right) + \frac{1}{2} \left(h^{1\,e}{}_{c;a;e} + h^{1\,e}{}_{a;c;e} - h^{1}{}_{ca}{}^{;e}{}_{;e}\right)\right)\right) + \\
& \frac{1}{2} \left(h^{1}{}_{da} R_{bc} + g_{da} \left(\frac{1}{2} \left(-h^{1\,e}{}_{e;c;b} - h^{1\,e}{}_{c;e;b} + h^{1}{}_{ce}{}^{;e}{}_{;b}\right) + \frac{1}{2} \left(h^{1\,e}{}_{c;b;e} + h^{1\,e}{}_{b;c;e} - h^{1}{}_{cb}{}^{;e}{}_{;e}\right)\right)\right) + \\
& \frac{1}{2} \left(h^{1}{}_{cb} R_{ad} + g_{cb} \left(\frac{1}{2} \left(-h^{1\,e}{}_{e;d;a} - h^{1\,e}{}_{d;e;a} + h^{1}{}_{de}{}^{;e}{}_{;a}\right) + \frac{1}{2} \left(h^{1\,e}{}_{d;a;e} + h^{1\,e}{}_{a;d;e} - h^{1}{}_{da}{}^{;e}{}_{;e}\right)\right)\right) + \\
& \frac{1}{2} \left(-h^{1}{}_{ca} R_{bd} - g_{ca} \left(\frac{1}{2} \left(-h^{1\,e}{}_{e;d;b} - h^{1\,e}{}_{d;e;b} + h^{1}{}_{de}{}^{;e}{}_{;b}\right) + \frac{1}{2} \left(h^{1\,e}{}_{d;b;e} + h^{1\,e}{}_{b;d;e} - h^{1}{}_{db}{}^{;e}{}_{;e}\right)\right)\right) + \\
& \frac{1}{6} \Bigg(-g_{da} h^{1}{}_{cb} R - g_{cb} h^{1}{}_{da} R - g_{cb} g_{da} \Bigg(-h^{1\,ef} R_{ef} + \\
& \qquad g^{ef} \left(\frac{1}{2} \left(-h^{1\,f1}{}_{f1;f;e} - h^{1\,f1}{}_{f;f1;e} + h^{1}{}_{ff1}{}^{;f1}{}_{;e}\right) + \frac{1}{2} \left(h^{1\,f1}{}_{f;e;f1} + h^{1\,f1}{}_{e;f;f1} - h^{1}{}_{fe}{}^{;f1}{}_{;f1}\right)\right)\Bigg)\Bigg) + \\
& \frac{1}{6} \Bigg(g_{db} h^{1}{}_{ca} R + g_{ca} h^{1}{}_{db} R + g_{ca} g_{db} \Bigg(-h^{1\,ef} R_{ef} + \\
& \qquad g^{ef} \left(\frac{1}{2} \left(-h^{1\,f1}{}_{f1;f;e} - h^{1\,f1}{}_{f;f1;e} + h^{1}{}_{ff1}{}^{;f1}{}_{;e}\right) + \frac{1}{2} \left(h^{1\,f1}{}_{f;e;f1} + h^{1\,f1}{}_{e;f;f1} - h^{1}{}_{fe}{}^{;f1}{}_{;f1}\right)\right)\Bigg)\Bigg)
\end{aligned}$$

The first−order perturbation applying directly `ExpandPerturbation` is also internally computed along the same lines:

In[148]:=

```
Perturbation[WeylCD[-a, -b, -c, -d], 1] // ExpandPerturbation
```

Out[148]=

$$\begin{aligned}
& h^{1}{}_{ed} R_{abc}{}^{e} + g_{ed} \left(\frac{1}{2} \left(-h^{1\,e}{}_{c;b;a} - h^{1\,e}{}_{b;c;a} + h^{1}{}_{cb}{}^{;e}{}_{;a}\right) + \frac{1}{2} \left(h^{1\,e}{}_{c;a;b} + h^{1\,e}{}_{a;c;b} - h^{1}{}_{ca}{}^{;e}{}_{;b}\right)\right) + \\
& \frac{1}{2} \left(-h^{1}{}_{db} R_{ac} - g_{db} \left(\frac{1}{2} \left(-h^{1\,e}{}_{e;c;a} - h^{1\,e}{}_{c;e;a} + h^{1}{}_{ce}{}^{;e}{}_{;a}\right) + \frac{1}{2} \left(h^{1\,e}{}_{c;a;e} + h^{1\,e}{}_{a;c;e} - h^{1}{}_{ca}{}^{;e}{}_{;e}\right)\right)\right) + \\
& \frac{1}{2} \left(h^{1}{}_{da} R_{bc} + g_{da} \left(\frac{1}{2} \left(-h^{1\,e}{}_{e;c;b} - h^{1\,e}{}_{c;e;b} + h^{1}{}_{ce}{}^{;e}{}_{;b}\right) + \frac{1}{2} \left(h^{1\,e}{}_{c;b;e} + h^{1\,e}{}_{b;c;e} - h^{1}{}_{cb}{}^{;e}{}_{;e}\right)\right)\right) + \\
& \frac{1}{2} \left(h^{1}{}_{cb} R_{ad} + g_{cb} \left(\frac{1}{2} \left(-h^{1\,e}{}_{e;d;a} - h^{1\,e}{}_{d;e;a} + h^{1}{}_{de}{}^{;e}{}_{;a}\right) + \frac{1}{2} \left(h^{1\,e}{}_{d;a;e} + h^{1\,e}{}_{a;d;e} - h^{1}{}_{da}{}^{;e}{}_{;e}\right)\right)\right) + \\
& \frac{1}{2} \left(-h^{1}{}_{ca} R_{bd} - g_{ca} \left(\frac{1}{2} \left(-h^{1\,e}{}_{e;d;b} - h^{1\,e}{}_{d;e;b} + h^{1}{}_{de}{}^{;e}{}_{;b}\right) + \frac{1}{2} \left(h^{1\,e}{}_{d;b;e} + h^{1\,e}{}_{b;d;e} - h^{1}{}_{db}{}^{;e}{}_{;e}\right)\right)\right) + \\
& \frac{1}{6} \Bigg(-g_{da} h^{1}{}_{cb} R - g_{cb} h^{1}{}_{da} R - g_{cb} g_{da} \Bigg(-h^{1\,ef} R_{ef} + \\
& \qquad g^{ef} \left(\frac{1}{2} \left(-h^{1\,f1}{}_{f1;f;e} - h^{1\,f1}{}_{f;f1;e} + h^{1}{}_{ff1}{}^{;f1}{}_{;e}\right) + \frac{1}{2} \left(h^{1\,f1}{}_{f;e;f1} + h^{1\,f1}{}_{e;f;f1} - h^{1}{}_{fe}{}^{;f1}{}_{;f1}\right)\right)\Bigg)\Bigg) + \\
& \frac{1}{6} \Bigg(g_{db} h^{1}{}_{ca} R + g_{ca} h^{1}{}_{db} R + g_{ca} g_{db} \Bigg(-h^{1\,ef} R_{ef} + \\
& \qquad g^{ef} \left(\frac{1}{2} \left(-h^{1\,f1}{}_{f1;f;e} - h^{1\,f1}{}_{f;f1;e} + h^{1}{}_{ff1}{}^{;f1}{}_{;e}\right) + \frac{1}{2} \left(h^{1\,f1}{}_{f;e;f1} + h^{1\,f1}{}_{e;f;f1} - h^{1}{}_{fe}{}^{;f1}{}_{;f1}\right)\right)\Bigg)\Bigg)
\end{aligned}$$

Comparison:

```
In[149]:=
    % - %% // ToCanonical

        0.184012 Second

Out[149]=
    0
```

4. Perturbations of other objects. Densities

Usually we have other, user−defined, geometric objects defined on our manifold. For instance, matter fields. In this section we will see how to construct a tensor that is the perturbation of another one. We will also show that this can be done for densities as well.

Let us define a rank−3 tensor T and a rank−2 symmetric +1−density Π. Note that the characters of the indices at definition will be later privileged in the computation.

```
In[150]:=
    DefTensor[T[-a, b, c], M]
    DefTensor[Π[b, c], M, Symmetric[{b, c}], WeightOfTensor → AIndex]

        ** DefTensor: Defining tensor T[-a, b, c].

        ** DefTensor: Defining tensor Π[b, c].
```

We can also define some constant symbol, whose perturbation would be automatically zero

```
In[152]:=
    DefConstantSymbol[κ]

        ** DefConstantSymbol: Defining constant symbol κ.
```

We define another two tensors which will represent the perturbations of the previous tensors, and hence must have the same density character. The syntaxis is parallel to that of `DefTensor`:

```
In[153]:=
    DefTensorPerturbation[τ[LI[order], -a, b, c], T[-a, b, c], M]
    DefTensorPerturbation[P[LI[order], b, c],
     Π[b, c], M, Symmetric[{b, c}], WeightOfTensor → AIndex]

        ** DefTensor: Defining tensor τ[LI[order], -a, b, c].

        ** DefTensor: Defining tensor P[LI[order], b, c].
```

Now we have

```
In[155]:=
    Perturbation[T[-a, b, c], 3]

Out[155]=
```
$\tau^{3}{}_{a}{}^{bc}$

In[156]:=
```
Perturbation[Π[a, b], 2]
```
Out[156]=

$P^{2\,ab}$

Now we can do the perturbation of any expression containing these and the geometric tensors with any index positioning. When the indices are in the natural position, the perturbation will be directly computed

In[157]:=
```
Perturbation[T[-a, b, c] Π[d, e], 2]
```
Out[157]=

$P^{2\,de}\, T_a{}^{bc} + 2\, P^{1\,de}\, \tau^1{}_a{}^{bc} + \tilde{\Pi}^{de}\, \tau^2{}_a{}^{bc}$

But when they are in a different position we have to make use of `ExpandPerturbation`

In[158]:=
```
Perturbation[T[-a, b, -c] Π[-d, e], 2]
```
Out[158]=

$2\, \Delta[T_a{}^b{}_c]\, \Delta[\tilde{\Pi}_d{}^e] + \Delta^2[\tilde{\Pi}_d{}^e]\, T_a{}^b{}_c + \Delta^2[T_a{}^b{}_c]\, \tilde{\Pi}_d{}^e$

In[159]:=
```
% // ExpandPerturbation
```
Out[159]=

$T_a{}^b{}_c \left(2\, h^1{}_{df}\, P^{1\,fe} + g_{df}\, P^{2\,fe} + h^2{}_{df}\, \tilde{\Pi}^{fe}\right) + 2 \left(g_{df1}\, P^{1\,f1e} + h^1{}_{df1}\, \tilde{\Pi}^{f1e}\right) \left(h^1{}_{cf}\, T_a{}^{bf} + g_{cf}\, \tau^1{}_a{}^{bf}\right) + \tilde{\Pi}_d{}^e \left(h^2{}_{cf}\, T_a{}^{bf} + 2\, h^1{}_{cf}\, \tau^1{}_a{}^{bf} + g_{cf}\, \tau^2{}_a{}^{bf}\right)$

We can also do more difficult computations at higher order

In[160]:=
```
RicciCD[-a, b] Π[-b, d] +
 Π[-a, c] RiemannCD[-c, b, -b, d] + Π[-a, b] T[d, -b, c] T[-c, -e, e]
```
Out[160]=

$T_{ce}{}^e\, T^d{}_b{}^c\, \tilde{\Pi}_a{}^b - R_c{}^d\, \tilde{\Pi}_a{}^c + R_a{}^b\, \tilde{\Pi}_b{}^d$

In[161]:=
```
Perturbation[%, 3]
```
Out[161]=

$6\, \Delta[T_{ce}{}^e]\, \Delta[T^d{}_b{}^c]\, \Delta[\tilde{\Pi}_a{}^b] + 3\, \Delta[\tilde{\Pi}_b{}^d]\, \Delta^2[R_a{}^b] - 3\, \Delta[\tilde{\Pi}_a{}^c]\, \Delta^2[R_c{}^d] - 3\, \Delta[R_c{}^d]\, \Delta^2[\tilde{\Pi}_a{}^c] + 3\, \Delta[R_a{}^b]\, \Delta^2[\tilde{\Pi}_b{}^d] + \Delta^3[\tilde{\Pi}_b{}^d]\, R_a{}^b - \Delta^3[\tilde{\Pi}_a{}^c]\, R_c{}^d + 3\, \Delta[\tilde{\Pi}_a{}^b]\, \Delta^2[T^d{}_b{}^c]\, T_{ce}{}^e + 3\, \Delta[T^d{}_b{}^c]\, \Delta^2[\tilde{\Pi}_a{}^b]\, T_{ce}{}^e + 3\, \Delta[\tilde{\Pi}_a{}^b]\, \Delta^2[T_{ce}{}^e]\, T^d{}_b{}^c + 3\, \Delta[T_{ce}{}^e]\, \Delta^2[\tilde{\Pi}_a{}^b]\, T^d{}_b{}^c + \Delta^3[\tilde{\Pi}_a{}^b]\, T_{ce}{}^e\, T^d{}_b{}^c + 3\, \Delta[T^d{}_b{}^c]\, \Delta^2[T_{ce}{}^e]\, \tilde{\Pi}_a{}^b + 3\, \Delta[T_{ce}{}^e]\, \Delta^2[T^d{}_b{}^c]\, \tilde{\Pi}_a{}^b + \Delta^3[T^d{}_b{}^c]\, T_{ce}{}^e\, \tilde{\Pi}_a{}^b + \Delta^3[T_{ce}{}^e]\, T^d{}_b{}^c\, \tilde{\Pi}_a{}^b - \Delta^3[R_c{}^d]\, \tilde{\Pi}_a{}^c + \Delta^3[R_a{}^b]\, \tilde{\Pi}_b{}^d$

In[162]:=
```
% // ExpandPerturbation // ContractMetric // ToCanonical
```
```
7.81249 Second
```

Out[162]=

$$\breve{P}^{3\,db}\,R_{ab} + 3\,h^{2\,db}\,\breve{P}^{1}{}_{a}{}^{c}\,R_{bc} + 3\,h^{1\,db}\,\breve{P}^{2}{}_{a}{}^{c}\,R_{bc} - 6\,h^{1}{}_{b}{}^{c}\,h^{1\,db}\,\breve{P}^{1}{}_{a}{}^{e}\,R_{ce} + 6\,h^{1}{}_{a}{}^{b}\,h^{1\,dc}\,\breve{P}^{1}{}_{b}{}^{e}\,R_{ce} -$$
$$\breve{P}^{3}{}_{a}{}^{b}\,R^{d}{}_{b} - 3\,h^{2}{}_{a}{}^{b}\,\breve{P}^{1}{}_{b}{}^{c}\,R^{d}{}_{c} - 3\,h^{1}{}_{a}{}^{b}\,\breve{P}^{2}{}_{b}{}^{c}\,R^{d}{}_{c} + 3\,h^{2}{}_{b}{}^{f}\,\breve{P}^{1}{}_{af}\,T_{c}{}^{e}{}_{e}\,T^{dbc} + 3\,h^{2}{}_{a}{}^{f}\,\breve{P}^{1}{}_{bf}\,T_{c}{}^{e}{}_{e}\,T^{dbc} +$$
$$6\,h^{1}{}_{a}{}^{f}\,h^{1}{}_{b}{}^{f1}\,\breve{P}^{1}{}_{ff1}\,T_{c}{}^{e}{}_{e}\,T^{dbc} + 3\,h^{1}{}_{b}{}^{f}\,\breve{P}^{2}{}_{af}\,T_{c}{}^{e}{}_{e}\,T^{dbc} + 3\,h^{1}{}_{a}{}^{f}\,\breve{P}^{2}{}_{bf}\,T_{c}{}^{e}{}_{e}\,T^{dbc} + \breve{P}^{3}{}_{ab}\,T_{c}{}^{e}{}_{e}\,T^{dbc} +$$
$$3\,h^{2}{}_{ef}\,\breve{P}^{1}{}_{ab}\,T_{c}{}^{ef}\,T^{dbc} + 6\,h^{1}{}_{b}{}^{f1}\,h^{1}{}_{ef}\,\breve{P}^{1}{}_{af1}\,T_{c}{}^{ef}\,T^{dbc} + 6\,h^{1}{}_{a}{}^{f1}\,h^{1}{}_{ef}\,\breve{P}^{1}{}_{bf1}\,T_{c}{}^{ef}\,T^{dbc} + 3\,h^{1}{}_{ef}\,\breve{P}^{2}{}_{ab}\,T_{c}{}^{ef}\,T^{dbc} -$$
$$6\,h^{1\,d}{}_{b}\,h^{1}{}_{ff1}\,\breve{P}^{1}{}_{ac}\,T^{bce}\,T_{e}{}^{ff1} + 6\,h^{1\,df1}\,h^{1}{}_{ef1}\,\breve{P}^{1}{}_{af}\,T^{bc}{}_{c}\,T^{ef}{}_{b} - 3\,h^{2\,d}{}_{e}\,\breve{P}^{1}{}_{af}\,T^{bc}{}_{c}\,T^{ef}{}_{b} -$$
$$6\,h^{1\,d}{}_{e}\,h^{1}{}_{f}{}^{f1}\,\breve{P}^{1}{}_{af1}\,T^{bc}{}_{c}\,T^{ef}{}_{b} - 6\,h^{1}{}_{a}{}^{f1}\,h^{1\,d}{}_{e}\,\breve{P}^{1}{}_{ff1}\,T^{bc}{}_{c}\,T^{ef}{}_{b} - 3\,h^{1\,d}{}_{e}\,\breve{P}^{2}{}_{af}\,T^{bc}{}_{c}\,T^{ef}{}_{b} +$$
$$h^{3}{}_{ef}\,T_{c}{}^{ef}\,T^{dbc}\,\tilde{\Pi}_{ab} + h^{3\,db}\,R_{b}{}^{c}\,\tilde{\Pi}_{ac} + 6\,h^{1}{}_{bf10}\,h^{1\,df10}\,h^{1}{}_{ff1}\,T^{bce}\,T_{e}{}^{ff1}\,\tilde{\Pi}_{ac} - 3\,h^{1}{}_{ff1}\,h^{2\,d}{}_{b}\,T^{bce}\,T_{e}{}^{ff1}\,\tilde{\Pi}_{ac} -$$
$$3\,h^{1\,d}{}_{b}\,h^{2}{}_{ff1}\,T^{bce}\,T_{e}{}^{ff1}\,\tilde{\Pi}_{ac} - 3\,h^{1\,db}\,h^{2}{}_{b}{}^{c}\,R_{c}{}^{e}\,\tilde{\Pi}_{ae} - 3\,h^{1}{}_{b}{}^{c}\,h^{2\,db}\,R_{c}{}^{e}\,\tilde{\Pi}_{ae} + 6\,h^{1}{}_{b}{}^{c}\,h^{1}{}_{c}{}^{e}\,h^{1\,db}\,R_{e}{}^{f}\,\tilde{\Pi}_{af} +$$
$$h^{3}{}_{b}{}^{f}\,T_{c}{}^{e}{}_{e}\,T^{dbc}\,\tilde{\Pi}_{af} - 6\,h^{1\,df1}\,h^{1}{}_{e}{}^{f10}\,h^{1}{}_{f1f10}\,T^{bc}{}_{c}\,T^{ef}{}_{b}\,\tilde{\Pi}_{af} + 3\,h^{1}{}_{ef1}\,h^{2\,df1}\,T^{bc}{}_{c}\,T^{ef}{}_{b}\,\tilde{\Pi}_{af} +$$
$$3\,h^{1\,df1}\,h^{2}{}_{ef1}\,T^{bc}{}_{c}\,T^{ef}{}_{b}\,\tilde{\Pi}_{af} - h^{3\,d}{}_{e}\,T^{bc}{}_{c}\,T^{ef}{}_{b}\,\tilde{\Pi}_{af} + 3\,h^{1}{}_{ef}\,h^{2}{}_{b}{}^{f1}\,T_{c}{}^{ef}\,T^{dbc}\,\tilde{\Pi}_{af1} +$$
$$3\,h^{1}{}_{b}{}^{f1}\,h^{2}{}_{ef}\,T_{c}{}^{ef}\,T^{dbc}\,\tilde{\Pi}_{af1} - 3\,h^{1}{}_{f}{}^{f1}\,h^{2\,d}{}_{e}\,T^{bc}{}_{c}\,T^{ef}{}_{b}\,\tilde{\Pi}_{af1} - 3\,h^{1\,d}{}_{e}\,h^{2}{}_{f}{}^{f1}\,T^{bc}{}_{c}\,T^{ef}{}_{b}\,\tilde{\Pi}_{af1} -$$
$$6\,h^{1}{}_{c}{}^{f10}\,h^{1\,d}{}_{b}\,h^{1}{}_{ff1}\,T^{bce}\,T_{e}{}^{ff1}\,\tilde{\Pi}_{af10} + 6\,h^{1\,df1}\,h^{1}{}_{ef1}\,h^{1}{}_{f}{}^{f10}\,T^{bc}{}_{c}\,T^{ef}{}_{b}\,\tilde{\Pi}_{af10} - h^{3}{}_{a}{}^{b}\,R^{dc}\,\tilde{\Pi}_{bc} +$$
$$3\,h^{1\,dc}\,h^{2}{}_{a}{}^{b}\,R_{c}{}^{e}\,\tilde{\Pi}_{be} + 3\,h^{1}{}_{a}{}^{b}\,h^{2\,dc}\,R_{c}{}^{e}\,\tilde{\Pi}_{be} - 6\,h^{1}{}_{a}{}^{b}\,h^{1}{}_{c}{}^{e}\,h^{1\,dc}\,R_{e}{}^{f}\,\tilde{\Pi}_{bf} + h^{3}{}_{a}{}^{f}\,T_{c}{}^{e}{}_{e}\,T^{dbc}\,\tilde{\Pi}_{bf} +$$
$$3\,h^{1}{}_{ef}\,h^{2}{}_{a}{}^{f1}\,T_{c}{}^{ef}\,T^{dbc}\,\tilde{\Pi}_{bf1} + 3\,h^{1}{}_{a}{}^{f1}\,h^{2}{}_{ef}\,T_{c}{}^{ef}\,T^{dbc}\,\tilde{\Pi}_{bf1} - 6\,h^{1}{}_{a}{}^{f10}\,h^{1\,d}{}_{b}\,h^{1}{}_{ff1}\,T^{bce}\,T_{e}{}^{ff1}\,\tilde{\Pi}_{cf10} +$$
$$3\,h^{1}{}_{b}{}^{f1}\,h^{2}{}_{a}{}^{f}\,T_{c}{}^{e}{}_{e}\,T^{dbc}\,\tilde{\Pi}_{ff1} + 3\,h^{1}{}_{a}{}^{f}\,h^{2}{}_{b}{}^{f1}\,T_{c}{}^{e}{}_{e}\,T^{dbc}\,\tilde{\Pi}_{ff1} + 6\,h^{1}{}_{a}{}^{f1}\,h^{1\,df10}\,h^{1}{}_{ef10}\,T^{bc}{}_{c}\,T^{ef}{}_{b}\,\tilde{\Pi}_{ff1} -$$
$$3\,h^{1\,d}{}_{e}\,h^{2}{}_{a}{}^{f1}\,T^{bc}{}_{c}\,T^{ef}{}_{b}\,\tilde{\Pi}_{ff1} - 3\,h^{1}{}_{a}{}^{f1}\,h^{2\,d}{}_{e}\,T^{bc}{}_{c}\,T^{ef}{}_{b}\,\tilde{\Pi}_{ff1} + 6\,h^{1}{}_{a}{}^{f1}\,h^{1}{}_{b}{}^{f10}\,h^{1}{}_{ef}\,T_{c}{}^{ef}\,T^{dbc}\,\tilde{\Pi}_{f1f10} -$$
$$6\,h^{1}{}_{a}{}^{f1}\,h^{1\,d}{}_{e}\,h^{1}{}_{f}{}^{f10}\,T^{bc}{}_{c}\,T^{ef}{}_{b}\,\tilde{\Pi}_{f1f10} + 6\,h^{1}{}_{b}{}^{f}\,\breve{P}^{1}{}_{af}\,T^{dbc}\,\tau^{1}{}_{c}{}^{e}{}_{e} + 6\,h^{1}{}_{a}{}^{f}\,\breve{P}^{1}{}_{bf}\,T^{dbc}\,\tau^{1}{}_{c}{}^{e}{}_{e} + 3\,\breve{P}^{2}{}_{ab}\,T^{dbc}\,\tau^{1}{}_{c}{}^{e}{}_{e} +$$
$$3\,h^{2}{}_{b}{}^{f}\,T^{dbc}\,\tilde{\Pi}_{af}\,\tau^{1}{}_{c}{}^{e}{}_{e} + 3\,h^{2}{}_{a}{}^{f}\,T^{dbc}\,\tilde{\Pi}_{bf}\,\tau^{1}{}_{c}{}^{e}{}_{e} + 6\,h^{1}{}_{a}{}^{f}\,h^{1}{}_{b}{}^{f1}\,T^{dbc}\,\tilde{\Pi}_{ff1}\,\tau^{1}{}_{c}{}^{e}{}_{e} + 6\,h^{1}{}_{ef}\,\breve{P}^{1}{}_{ab}\,T^{dbc}\,\tau^{1}{}_{c}{}^{ef} +$$
$$3\,h^{2}{}_{ef}\,T^{dbc}\,\tilde{\Pi}_{ab}\,\tau^{1}{}_{c}{}^{ef} + 6\,h^{1}{}_{b}{}^{f1}\,h^{1}{}_{ef}\,T^{dbc}\,\tilde{\Pi}_{af1}\,\tau^{1}{}_{c}{}^{ef} + 6\,h^{1}{}_{a}{}^{f1}\,h^{1}{}_{ef}\,T^{dbc}\,\tilde{\Pi}_{bf1}\,\tau^{1}{}_{c}{}^{ef} +$$
$$6\,\breve{P}^{1}{}_{ab}\,\tau^{1}{}_{c}{}^{e}{}_{e}\,\tau^{1\,dbc} + 6\,h^{1}{}_{b}{}^{f}\,\tilde{\Pi}_{af}\,\tau^{1}{}_{c}{}^{e}{}_{e}\,\tau^{1\,dbc} + 6\,h^{1}{}_{a}{}^{f}\,\tilde{\Pi}_{bf}\,\tau^{1}{}_{c}{}^{e}{}_{e}\,\tau^{1\,dbc} + 6\,h^{1}{}_{ef}\,\tilde{\Pi}_{ab}\,\tau^{1}{}_{c}{}^{ef}\,\tau^{1\,dbc} +$$
$$6\,h^{1}{}_{e}{}^{f}\,\breve{P}^{1}{}_{af}\,T^{bc}{}_{c}\,\tau^{1\,de}{}_{b} + 6\,h^{1}{}_{a}{}^{f}\,\breve{P}^{1}{}_{ef}\,T^{bc}{}_{c}\,\tau^{1\,de}{}_{b} + 3\,\breve{P}^{2}{}_{ae}\,T^{bc}{}_{c}\,\tau^{1\,de}{}_{b} + 3\,h^{2}{}_{e}{}^{f}\,T^{bc}{}_{c}\,\tilde{\Pi}_{af}\,\tau^{1\,de}{}_{b} +$$
$$3\,h^{2}{}_{a}{}^{f}\,T^{bc}{}_{c}\,\tilde{\Pi}_{ef}\,\tau^{1\,de}{}_{b} + 6\,h^{1}{}_{a}{}^{f}\,h^{1}{}_{e}{}^{f1}\,T^{bc}{}_{c}\,\tilde{\Pi}_{ff1}\,\tau^{1\,de}{}_{b} + 6\,h^{1}{}_{ce}\,\breve{P}^{1}{}_{af}\,T^{bce}\,\tau^{1\,df}{}_{b} + 3\,h^{2}{}_{ce}\,T^{bce}\,\tilde{\Pi}_{af}\,\tau^{1\,df}{}_{b} +$$
$$6\,h^{1}{}_{ce}\,h^{1}{}_{f}{}^{f1}\,T^{bce}\,\tilde{\Pi}_{af1}\,\tau^{1\,df}{}_{b} + 6\,h^{1}{}_{a}{}^{f1}\,h^{1}{}_{ce}\,T^{bce}\,\tilde{\Pi}_{ff1}\,\tau^{1\,df}{}_{b} - 6\,h^{1\,d}{}_{b}\,\breve{P}^{1}{}_{ac}\,T^{bce}\,\tau^{1}{}_{e}{}^{f}{}_{f} +$$
$$6\,h^{1}{}_{bf1}\,h^{1\,df1}\,T^{bce}\,\tilde{\Pi}_{ac}\,\tau^{1}{}_{e}{}^{f}{}_{f} - 3\,h^{2\,d}{}_{b}\,T^{bce}\,\tilde{\Pi}_{ac}\,\tau^{1}{}_{e}{}^{f}{}_{f} - 6\,h^{1}{}_{c}{}^{f1}\,h^{1\,d}{}_{b}\,T^{bce}\,\tilde{\Pi}_{af1}\,\tau^{1}{}_{e}{}^{f}{}_{f} -$$
$$6\,h^{1}{}_{a}{}^{f1}\,h^{1\,d}{}_{b}\,T^{bce}\,\tilde{\Pi}_{cf1}\,\tau^{1}{}_{e}{}^{f}{}_{f} - 6\,h^{1\,d}{}_{b}\,h^{1}{}_{ff1}\,T^{bce}\,\tilde{\Pi}_{ac}\,\tau^{1}{}_{e}{}^{ff1} - 6\,h^{1\,d}{}_{e}\,\breve{P}^{1}{}_{af}\,T^{bc}{}_{c}\,\tau^{1\,ef}{}_{b} +$$
$$6\,h^{1\,df1}\,h^{1}{}_{ef1}\,T^{bc}{}_{c}\,\tilde{\Pi}_{af}\,\tau^{1\,ef}{}_{b} - 3\,h^{2\,d}{}_{e}\,T^{bc}{}_{c}\,\tilde{\Pi}_{af}\,\tau^{1\,ef}{}_{b} - 6\,h^{1\,d}{}_{e}\,h^{1}{}_{f}{}^{f1}\,T^{bc}{}_{c}\,\tilde{\Pi}_{af1}\,\tau^{1\,ef}{}_{b} -$$
$$6\,h^{1}{}_{a}{}^{f1}\,h^{1\,d}{}_{e}\,T^{bc}{}_{c}\,\tilde{\Pi}_{ff1}\,\tau^{1\,ef}{}_{b} - 6\,h^{1\,d}{}_{e}\,\tilde{\Pi}_{af}\,\tau^{1\,bc}{}_{c}\,\tau^{1\,ef}{}_{b} - 6\,h^{1}{}_{ce}\,h^{1\,d}{}_{f}\,T^{bce}\,\tilde{\Pi}_{af1}\,\tau^{1\,ff1}{}_{b} +$$
$$3\,\breve{P}^{1}{}_{ab}\,T^{dbc}\,\tau^{2}{}_{c}{}^{e}{}_{e} + 3\,h^{1}{}_{b}{}^{f}\,T^{dbc}\,\tilde{\Pi}_{af}\,\tau^{2}{}_{c}{}^{e}{}_{e} + 3\,h^{1}{}_{a}{}^{f}\,T^{dbc}\,\tilde{\Pi}_{bf}\,\tau^{2}{}_{c}{}^{e}{}_{e} + 3\,\tilde{\Pi}_{ab}\,\tau^{1\,dbc}\,\tau^{2}{}_{c}{}^{e}{}_{e} +$$
$$3\,h^{1}{}_{ef}\,T^{dbc}\,\tilde{\Pi}_{ab}\,\tau^{2}{}_{c}{}^{ef} + 3\,\tilde{\Pi}_{ab}\,\tau^{1}{}_{c}{}^{e}{}_{e}\,\tau^{2\,dbc} + 3\,\breve{P}^{1}{}_{ae}\,T^{bc}{}_{c}\,\tau^{2\,de}{}_{b} + 3\,h^{1}{}_{e}{}^{f}\,T^{bc}{}_{c}\,\tilde{\Pi}_{af}\,\tau^{2\,de}{}_{b} +$$
$$3\,h^{1}{}_{a}{}^{f}\,T^{bc}{}_{c}\,\tilde{\Pi}_{ef}\,\tau^{2\,de}{}_{b} + 3\,h^{1}{}_{ce}\,T^{bce}\,\tilde{\Pi}_{af}\,\tau^{2\,df}{}_{b} - 3\,h^{1\,d}{}_{b}\,T^{bce}\,\tilde{\Pi}_{ac}\,\tau^{2}{}_{e}{}^{f}{}_{f} - 3\,h^{1\,d}{}_{e}\,T^{bc}{}_{c}\,\tilde{\Pi}_{af}\,\tau^{2\,ef}{}_{b} +$$
$$T^{dbc}\,\tilde{\Pi}_{ab}\,\tau^{3}{}_{c}{}^{e}{}_{e} + T^{bc}{}_{c}\,\tilde{\Pi}_{ae}\,\tau^{3\,de}{}_{b} + 3\,h^{1\,bc}\,\breve{P}^{1\,de}\,h^{1}{}_{bc;e;a} + \frac{3}{2}\,h^{2\,bc}\,\tilde{\Pi}^{de}\,h^{1}{}_{bc;e;a} + \frac{3}{2}\,h^{1\,bc}\,\tilde{\Pi}^{de}\,h^{2}{}_{bc;e;a} -$$
$$3\,h^{1}{}_{b}{}^{e}\,h^{1\,bc}\,\tilde{\Pi}^{df}\,h^{1}{}_{ce;f;a} + \frac{3}{2}\,\breve{P}^{1\,db}\,h^{1\,ce}{}_{;a}\,h^{1}{}_{ce;b} + \frac{3}{4}\,\tilde{\Pi}^{db}\,h^{2\,ce}{}_{;a}\,h^{1}{}_{ce;b} + \frac{3}{4}\,\tilde{\Pi}^{db}\,h^{1\,ce}{}_{;a}\,h^{2}{}_{ce;b} -$$
$$\frac{3}{2}\,\breve{P}^{2\,db}\,h^{1\,c}{}_{c;a;b} - \frac{3}{2}\,\breve{P}^{1\,db}\,h^{2\,c}{}_{c;a;b} - \frac{1}{2}\,\tilde{\Pi}^{db}\,h^{3\,c}{}_{c;a;b} + \frac{3}{2}\,\breve{P}^{2}{}_{a}{}^{b}\,h^{1\,c}{}_{c}{}^{;d}{}_{;b} + \frac{3}{2}\,\breve{P}^{1}{}_{a}{}^{b}\,h^{2\,c}{}_{c}{}^{;d}{}_{;b} +$$
$$\frac{1}{2}\,\tilde{\Pi}_{a}{}^{b}\,h^{3\,c}{}_{c}{}^{;d}{}_{;b} + 3\,h^{1\,bc}\,\tilde{\Pi}^{de}\,h^{1}{}_{e}{}^{f}{}_{;a}\,h^{1}{}_{bf;c} - 3\,h^{1\,bc}\,\tilde{\Pi}^{de}\,h^{1}{}_{a}{}^{f}{}_{;b}\,h^{1}{}_{ef;c} + 3\,h^{1\,bc}\,\tilde{\Pi}_{a}{}^{e}\,h^{1\,df}{}_{;b}\,h^{1}{}_{ef;c} +$$
$$\frac{3}{2}\,h^{1\,db}\,\tilde{\Pi}_{a}{}^{c}\,h^{1\,ef}{}_{;b}\,h^{1}{}_{ef;c} + \frac{3}{2}\,\breve{P}^{1\,db}\,h^{1}{}_{b}{}^{c}{}_{;a}\,h^{1\,e}{}_{e;c} + \frac{3}{4}\,\tilde{\Pi}^{db}\,h^{2}{}_{b}{}^{c}{}_{;a}\,h^{1\,e}{}_{e;c} + \frac{3}{2}\,\breve{P}^{1\,db}\,h^{1}{}_{a}{}^{c}{}_{;b}\,h^{1\,e}{}_{e;c} -$$
$$\frac{3}{2}\,\breve{P}^{1}{}_{a}{}^{b}\,h^{1\,dc}{}_{;b}\,h^{1\,e}{}_{e;c} + \frac{3}{4}\,\tilde{\Pi}^{db}\,h^{2}{}_{a}{}^{c}{}_{;b}\,h^{1\,e}{}_{e;c} - \frac{3}{4}\,\tilde{\Pi}_{a}{}^{b}\,h^{2\,dc}{}_{;b}\,h^{1\,e}{}_{e;c} - \frac{3}{2}\,h^{1\,bc}\,\tilde{\Pi}^{de}\,h^{1}{}_{be;a}\,h^{1\,f}{}_{f;c} +$$
$$\frac{3}{2}\,h^{1\,bc}\,\tilde{\Pi}^{de}\,h^{1}{}_{ae;b}\,h^{1\,f}{}_{f;c} - \frac{3}{2}\,h^{1\,bc}\,\tilde{\Pi}_{a}{}^{e}\,h^{1\,d}{}_{e;b}\,h^{1\,f}{}_{f;c} + \frac{3}{4}\,\tilde{\Pi}^{db}\,h^{1}{}_{b}{}^{c}{}_{;a}\,h^{2\,e}{}_{e;c} + \frac{3}{4}\,\tilde{\Pi}^{db}\,h^{1}{}_{a}{}^{c}{}_{;b}\,h^{2\,e}{}_{e;c} -$$
$$\frac{3}{4}\,\tilde{\Pi}_{a}{}^{b}\,h^{1\,dc}{}_{;b}\,h^{2\,e}{}_{e;c} + \frac{3}{2}\,\breve{P}^{2\,db}\,h^{1}{}_{b}{}^{c}{}_{;a;c} - 3\,h^{1\,bc}\,\breve{P}^{1\,de}\,h^{1}{}_{be;a;c} - \frac{3}{2}\,h^{2\,bc}\,\tilde{\Pi}^{de}\,h^{1}{}_{be;a;c} + \frac{3}{2}\,\breve{P}^{1\,db}\,h^{2}{}_{b}{}^{c}{}_{;a;c} -$$
$$\frac{3}{2}\,h^{1\,bc}\,\tilde{\Pi}^{de}\,h^{2}{}_{be;a;c} + \frac{1}{2}\,\tilde{\Pi}^{db}\,h^{3}{}_{b}{}^{c}{}_{;a;c} + \frac{3}{2}\,\breve{P}^{2\,db}\,h^{1}{}_{a}{}^{c}{}_{;b;c} + 3\,h^{1\,bc}\,\breve{P}^{1\,de}\,h^{1}{}_{ae;b;c} + \frac{3}{2}\,h^{2\,bc}\,\tilde{\Pi}^{de}\,h^{1}{}_{ae;b;c} -$$

$$\frac{3}{2}\tilde{P}^{2}{}_{a}{}^{b}\,h^{1\,dc}{}_{;b;c}-3\,h^{1\,bc}\,\tilde{P}^{1}{}_{a}{}^{e}\,h^{1\,d}{}_{e;b;c}-\frac{3}{2}h^{2\,bc}\,\tilde{\Pi}_{a}{}^{e}\,h^{1\,d}{}_{e;b;c}-3\,h^{1\,db}\,\tilde{P}^{1}{}_{a}{}^{c}\,h^{1\,e}{}_{e;b;c}-$$

$$\frac{3}{2}h^{2\,db}\,\tilde{\Pi}_{a}{}^{c}\,h^{1\,e}{}_{e;b;c}+\frac{3}{2}\tilde{P}^{1\,db}\,h^{2}{}_{a}{}^{c}{}_{;b;c}+\frac{3}{2}h^{1\,bc}\,\tilde{\Pi}^{de}\,h^{2}{}_{ae;b;c}-\frac{3}{2}\tilde{P}^{1}{}_{a}{}^{b}\,h^{2\,dc}{}_{;b;c}-\frac{3}{2}h^{1\,bc}\,\tilde{\Pi}_{a}{}^{e}\,h^{2\,d}{}_{e;b;c}-$$

$$\frac{3}{2}h^{1\,db}\,\tilde{\Pi}_{a}{}^{c}\,h^{2\,e}{}_{e;b;c}+\frac{1}{2}\tilde{\Pi}^{db}\,h^{3}{}_{a}{}^{c}{}_{;b;c}-\frac{1}{2}\tilde{\Pi}_{a}{}^{b}\,h^{3\,dc}{}_{;b;c}-\frac{3}{2}\tilde{P}^{2\,db}\,h^{1}{}_{ab}{}^{;c}{}_{;c}+\frac{3}{2}\tilde{P}^{2}{}_{a}{}^{b}\,h^{1\,d}{}_{b}{}^{;c}{}_{;c}-$$

$$\frac{3}{2}\tilde{P}^{1\,db}\,h^{2}{}_{ab}{}^{;c}{}_{;c}+\frac{3}{2}\tilde{P}^{1}{}_{a}{}^{b}\,h^{2\,d}{}_{b}{}^{;c}{}_{;c}-\frac{1}{2}\tilde{\Pi}^{db}\,h^{3}{}_{ab}{}^{;c}{}_{;c}+\frac{1}{2}\tilde{\Pi}_{a}{}^{b}\,h^{3\,d}{}_{b}{}^{;c}{}_{;c}-\frac{3}{2}\tilde{P}^{2}{}_{a}{}^{b}\,h^{1}{}_{b}{}^{c;d}{}_{;c}+$$

$$3\,h^{1\,bc}\,\tilde{P}^{1}{}_{a}{}^{e}\,h^{1}{}_{be}{}^{;d}{}_{;c}+\frac{3}{2}h^{2\,bc}\,\tilde{\Pi}_{a}{}^{e}\,h^{1}{}_{be}{}^{;d}{}_{;c}+3\,h^{1}{}_{a}{}^{b}\,\tilde{P}^{1}{}_{b}{}^{c}\,h^{1\,e}{}_{e}{}^{;d}{}_{;c}+\frac{3}{2}h^{2}{}_{a}{}^{b}\,\tilde{\Pi}_{b}{}^{c}\,h^{1\,e}{}_{e}{}^{;d}{}_{;c}-$$

$$\frac{3}{2}\tilde{P}^{1}{}_{a}{}^{b}\,h^{2}{}_{b}{}^{c;d}{}_{;c}+\frac{3}{2}h^{1\,bc}\,\tilde{\Pi}_{a}{}^{e}\,h^{2}{}_{be}{}^{;d}{}_{;c}+\frac{3}{2}h^{1}{}_{a}{}^{b}\,\tilde{\Pi}_{b}{}^{c}\,h^{2\,e}{}_{e}{}^{;d}{}_{;c}-\frac{1}{2}\tilde{\Pi}_{a}{}^{b}\,h^{3}{}_{b}{}^{c;d}{}_{;c}-3\,h^{1\,bc}\,\tilde{P}^{1\,de}\,h^{1}{}_{ab;e;c}-$$

$$\frac{3}{2}h^{2\,bc}\,\tilde{\Pi}^{de}\,h^{1}{}_{ab;e;c}+3\,h^{1\,bc}\,\tilde{P}^{1}{}_{a}{}^{e}\,h^{1\,d}{}_{b;e;c}+\frac{3}{2}h^{2\,bc}\,\tilde{\Pi}_{a}{}^{e}\,h^{1\,d}{}_{b;e;c}-\frac{3}{2}h^{1\,bc}\,\tilde{\Pi}^{de}\,h^{2}{}_{ab;e;c}+$$

$$\frac{3}{2}h^{1\,bc}\,\tilde{\Pi}_{a}{}^{e}\,h^{2\,d}{}_{b;e;c}-\frac{3}{2}\tilde{P}^{1\,db}\,h^{1\,e}{}_{e;c}\,h^{1}{}_{ab}{}^{;c}-\frac{3}{4}\tilde{\Pi}^{db}\,h^{2\,e}{}_{e;c}\,h^{1}{}_{ab}{}^{;c}+\frac{3}{2}\tilde{P}^{1}{}_{a}{}^{b}\,h^{1\,e}{}_{e;c}\,h^{1\,d}{}_{b}{}^{;c}+$$

$$\frac{3}{4}\tilde{\Pi}_{a}{}^{b}\,h^{2\,e}{}_{e;c}\,h^{1\,d}{}_{b}{}^{;c}-\frac{3}{4}\tilde{\Pi}^{db}\,h^{1\,e}{}_{e;c}\,h^{2}{}_{ab}{}^{;c}+\frac{3}{4}\tilde{\Pi}_{a}{}^{b}\,h^{1\,e}{}_{e;c}\,h^{2\,d}{}_{b}{}^{;c}-\frac{3}{2}\tilde{P}^{1}{}_{a}{}^{b}\,h^{1\,e}{}_{e;c}\,h^{1}{}_{b}{}^{c;d}-$$

$$\frac{3}{4}\tilde{\Pi}_{a}{}^{b}\,h^{2\,e}{}_{e;c}\,h^{1}{}_{b}{}^{c;d}+\frac{3}{2}h^{1\,bc}\,\tilde{\Pi}_{a}{}^{e}\,h^{1\,f}{}_{f;c}\,h^{1}{}_{be}{}^{;d}-\frac{3}{2}\tilde{P}^{1}{}_{a}{}^{b}\,h^{1}{}_{ce;b}\,h^{1\,ce;d}-\frac{3}{4}\tilde{\Pi}_{a}{}^{b}\,h^{2}{}_{ce;b}\,h^{1\,ce;d}-$$

$$3\,h^{1\,bc}\,\tilde{\Pi}_{a}{}^{e}\,h^{1}{}_{bf;c}\,h^{1}{}_{e}{}^{f;d}-\frac{3}{2}h^{1}{}_{a}{}^{b}\,\tilde{\Pi}_{b}{}^{c}\,h^{1}{}_{ef;c}\,h^{1\,ef;d}-\frac{3}{4}\tilde{\Pi}_{a}{}^{b}\,h^{1\,e}{}_{e;c}\,h^{2}{}_{b}{}^{c;d}-\frac{3}{4}\tilde{\Pi}_{a}{}^{b}\,h^{1}{}_{ce;b}\,h^{2\,ce;d}-$$

$$\frac{3}{2}h^{1\,bc}\,\tilde{\Pi}^{de}\,h^{1\,f}{}_{f;c}\,h^{1}{}_{ab;e}+3\,h^{1\,bc}\,\tilde{\Pi}^{de}\,h^{1}{}_{bf;c}\,h^{1}{}_{a}{}^{f}{}_{;e}-3\,\tilde{P}^{1\,db}\,h^{1}{}_{b}{}^{c}{}_{;a}\,h^{1}{}_{c}{}^{e}{}_{;e}-\frac{3}{2}\tilde{\Pi}^{db}\,h^{2}{}_{b}{}^{c}{}_{;a}\,h^{1}{}_{c}{}^{e}{}_{;e}-$$

$$3\,\tilde{P}^{1\,db}\,h^{1}{}_{a}{}^{c}{}_{;b}\,h^{1}{}_{c}{}^{e}{}_{;e}+3\,\tilde{P}^{1}{}_{a}{}^{b}\,h^{1\,dc}{}_{;b}\,h^{1}{}_{c}{}^{e}{}_{;e}-\frac{3}{2}\tilde{\Pi}^{db}\,h^{2}{}_{a}{}^{c}{}_{;b}\,h^{1}{}_{c}{}^{e}{}_{;e}+\frac{3}{2}\tilde{\Pi}_{a}{}^{b}\,h^{2\,dc}{}_{;b}\,h^{1}{}_{c}{}^{e}{}_{;e}+$$

$$3\,\tilde{P}^{1\,db}\,h^{1}{}_{ab}{}^{;c}\,h^{1}{}_{c}{}^{e}{}_{;e}-3\,\tilde{P}^{1}{}_{a}{}^{b}\,h^{1\,d}{}_{b}{}^{;c}\,h^{1}{}_{c}{}^{e}{}_{;e}+\frac{3}{2}\tilde{\Pi}^{db}\,h^{2}{}_{ab}{}^{;c}\,h^{1}{}_{c}{}^{e}{}_{;e}-\frac{3}{2}\tilde{\Pi}_{a}{}^{b}\,h^{2\,d}{}_{b}{}^{;c}\,h^{1}{}_{c}{}^{e}{}_{;e}+$$

$$3\,\tilde{P}^{1}{}_{a}{}^{b}\,h^{1}{}_{b}{}^{c;d}\,h^{1}{}_{c}{}^{e}{}_{;e}+\frac{3}{2}\tilde{\Pi}_{a}{}^{b}\,h^{2}{}_{b}{}^{c;d}\,h^{1}{}_{c}{}^{e}{}_{;e}-3\,h^{1\,bc}\,\tilde{\Pi}^{de}\,h^{1}{}_{b}{}^{f}{}_{;a}\,h^{1}{}_{cf;e}+3\,h^{1\,bc}\,\tilde{\Pi}_{a}{}^{e}\,h^{1}{}_{b}{}^{f;d}\,h^{1}{}_{cf;e}+$$

$$\frac{3}{2}h^{1\,bc}\,\tilde{\Pi}_{a}{}^{e}\,h^{1\,f}{}_{f;c}\,h^{1\,d}{}_{b;e}-3\,h^{1\,bc}\,\tilde{\Pi}_{a}{}^{e}\,h^{1}{}_{bf;c}\,h^{1\,df}{}_{;e}+\frac{3}{2}h^{1\,db}\,\tilde{\Pi}_{a}{}^{c}\,h^{1}{}_{c}{}^{e}{}_{;b}\,h^{1\,f}{}_{f;e}+$$

$$\frac{3}{2}h^{1\,db}\,\tilde{\Pi}_{a}{}^{c}\,h^{1}{}_{b}{}^{e}{}_{;c}\,h^{1\,f}{}_{f;e}-\frac{3}{2}h^{1}{}_{a}{}^{b}\,\tilde{\Pi}_{b}{}^{c}\,h^{1\,de}{}_{;c}\,h^{1\,f}{}_{f;e}-\frac{3}{2}h^{1}{}_{a}{}^{b}\,\tilde{\Pi}_{b}{}^{c}\,h^{1}{}_{c}{}^{e;d}\,h^{1\,f}{}_{f;e}-$$

$$\frac{3}{2}\tilde{\Pi}^{db}\,h^{1}{}_{b}{}^{c}{}_{;a}\,h^{2}{}_{c}{}^{e}{}_{;e}-\frac{3}{2}\tilde{\Pi}^{db}\,h^{1}{}_{a}{}^{c}{}_{;b}\,h^{2}{}_{c}{}^{e}{}_{;e}+\frac{3}{2}\tilde{\Pi}_{a}{}^{b}\,h^{1\,dc}{}_{;b}\,h^{2}{}_{c}{}^{e}{}_{;e}+\frac{3}{2}\tilde{\Pi}^{db}\,h^{1}{}_{ab}{}^{;c}\,h^{2}{}_{c}{}^{e}{}_{;e}-$$

$$\frac{3}{2}\tilde{\Pi}_{a}{}^{b}\,h^{1\,d}{}_{b}{}^{;c}\,h^{2}{}_{c}{}^{e}{}_{;e}+\frac{3}{2}\tilde{\Pi}_{a}{}^{b}\,h^{1}{}_{b}{}^{c;d}\,h^{2}{}_{c}{}^{e}{}_{;e}+3\,h^{1}{}_{b}{}^{e}\,h^{1\,bc}\,\tilde{\Pi}^{df}\,h^{1}{}_{cf;a;e}+3\,h^{1\,db}\,\tilde{P}^{1}{}_{a}{}^{c}\,h^{1}{}_{c}{}^{e}{}_{;b;e}+$$

$$\frac{3}{2}h^{2\,db}\,\tilde{\Pi}_{a}{}^{c}\,h^{1}{}_{c}{}^{e}{}_{;b;e}-3\,h^{1\,ce}\,h^{1\,db}\,\tilde{\Pi}_{a}{}^{f}\,h^{1}{}_{cf;b;e}+\frac{3}{2}h^{1\,db}\,\tilde{\Pi}_{a}{}^{c}\,h^{2}{}_{c}{}^{e}{}_{;b;e}-3\,h^{1}{}_{b}{}^{e}\,h^{1\,bc}\,\tilde{\Pi}^{df}\,h^{1}{}_{af;c;e}+$$

$$3\,h^{1\,db}\,\tilde{P}^{1}{}_{a}{}^{c}\,h^{1}{}_{b}{}^{e}{}_{;c;e}+\frac{3}{2}h^{2\,db}\,\tilde{\Pi}_{a}{}^{c}\,h^{1}{}_{b}{}^{e}{}_{;c;e}+3\,h^{1\,ce}\,h^{1\,db}\,\tilde{\Pi}_{a}{}^{f}\,h^{1}{}_{bf;c;e}-3\,h^{1}{}_{a}{}^{b}\,\tilde{P}^{1}{}_{b}{}^{c}\,h^{1\,de}{}_{;c;e}-$$

$$\frac{3}{2}h^{2}{}_{a}{}^{b}\,\tilde{\Pi}_{b}{}^{c}\,h^{1\,de}{}_{;c;e}+3\,h^{1}{}_{b}{}^{e}\,h^{1\,bc}\,\tilde{\Pi}_{a}{}^{f}\,h^{1\,d}{}_{f;c;e}-3\,h^{1}{}_{a}{}^{b}\,h^{1\,ce}\,\tilde{\Pi}_{b}{}^{f}\,h^{1\,d}{}_{f;c;e}+3\,h^{1}{}_{b}{}^{c}\,h^{1\,db}\,\tilde{\Pi}_{a}{}^{e}\,h^{1\,f}{}_{f;c;e}-$$

$$3\,h^{1}{}_{a}{}^{b}\,h^{1\,dc}\,\tilde{\Pi}_{b}{}^{e}\,h^{1\,f}{}_{f;c;e}+\frac{3}{2}h^{1\,db}\,\tilde{\Pi}_{a}{}^{c}\,h^{2}{}_{b}{}^{e}{}_{;c;e}-\frac{3}{2}h^{1}{}_{a}{}^{b}\,\tilde{\Pi}_{b}{}^{c}\,h^{2\,de}{}_{;c;e}-3\,h^{1\,bc}\,\tilde{P}^{1}{}_{a}{}^{e}\,h^{1}{}_{bc}{}^{;d}{}_{;e}-$$

$$\frac{3}{2}h^{2\,bc}\,\tilde{\Pi}_{a}{}^{e}\,h^{1}{}_{bc}{}^{;d}{}_{;e}-3\,h^{1}{}_{a}{}^{b}\,\tilde{P}^{1}{}_{b}{}^{c}\,h^{1}{}_{c}{}^{e;d}{}_{;e}-\frac{3}{2}h^{2}{}_{a}{}^{b}\,\tilde{\Pi}_{b}{}^{c}\,h^{1}{}_{c}{}^{e;d}{}_{;e}-3\,h^{1}{}_{b}{}^{e}\,h^{1\,bc}\,\tilde{\Pi}_{a}{}^{f}\,h^{1}{}_{cf}{}^{;d}{}_{;e}+$$

$$3\,h^{1}{}_{a}{}^{b}\,h^{1\,ce}\,\tilde{\Pi}_{b}{}^{f}\,h^{1}{}_{cf}{}^{;d}{}_{;e}-\frac{3}{2}h^{1\,bc}\,\tilde{\Pi}_{a}{}^{e}\,h^{2}{}_{bc}{}^{;d}{}_{;e}-\frac{3}{2}h^{1}{}_{a}{}^{b}\,\tilde{\Pi}_{b}{}^{c}\,h^{2}{}_{c}{}^{e;d}{}_{;e}-3\,h^{1\,db}\,\tilde{P}^{1}{}_{a}{}^{c}\,h^{1}{}_{bc}{}^{;e}{}_{;e}-$$

$$\frac{3}{2}h^{2\,db}\,\tilde{\Pi}_{a}{}^{c}\,h^{1}{}_{bc}{}^{;e}{}_{;e}+3\,h^{1}{}_{a}{}^{b}\,\tilde{P}^{1}{}_{b}{}^{c}\,h^{1\,d}{}_{c}{}^{;e}{}_{;e}+\frac{3}{2}h^{2}{}_{a}{}^{b}\,\tilde{\Pi}_{b}{}^{c}\,h^{1\,d}{}_{c}{}^{;e}{}_{;e}-\frac{3}{2}h^{1\,db}\,\tilde{\Pi}_{a}{}^{c}\,h^{2}{}_{bc}{}^{;e}{}_{;e}+$$

$$\frac{3}{2}h^{1}{}_{a}{}^{b}\,\tilde{\Pi}_{b}{}^{c}\,h^{2\,d}{}_{c}{}^{;e}{}_{;e}+3\,h^{1}{}_{b}{}^{e}\,h^{1\,bc}\,\tilde{\Pi}^{df}\,h^{1}{}_{ac;f;e}-3\,h^{1\,ce}\,h^{1\,db}\,\tilde{\Pi}_{a}{}^{f}\,h^{1}{}_{bc;f;e}-3\,h^{1}{}_{b}{}^{e}\,h^{1\,bc}\,\tilde{\Pi}_{a}{}^{f}\,h^{1\,d}{}_{c;f;e}+$$

$$3\,h^{1}{}_{a}{}^{b}\,h^{1\,ce}\,\tilde{\Pi}_{b}{}^{f}\,h^{1\,d}{}_{c;f;e}-3\,\tilde{P}^{1\,db}\,h^{1}{}_{be;c}\,h^{1}{}_{a}{}^{c;e}-\frac{3}{2}\tilde{\Pi}^{db}\,h^{2}{}_{be;c}\,h^{1}{}_{a}{}^{c;e}+3\,\tilde{P}^{1\,db}\,h^{1}{}_{bc;e}\,h^{1}{}_{a}{}^{c;e}+$$

$$\frac{3}{2}\tilde{\Pi}^{db}\,h^{2}{}_{bc;e}\,h^{1}{}_{a}{}^{c;e}-\frac{3}{2}h^{1\,db}\,\tilde{\Pi}_{a}{}^{c}\,h^{1\,f}{}_{f;e}\,h^{1}{}_{bc}{}^{;e}+\frac{3}{2}h^{1}{}_{a}{}^{b}\,\tilde{\Pi}_{b}{}^{c}\,h^{1\,f}{}_{f;e}\,h^{1\,d}{}_{c}{}^{;e}+3\,\tilde{P}^{1}{}_{a}{}^{b}\,h^{1}{}_{be;c}\,h^{1\,dc;e}+$$

$$\frac{3}{2}\tilde{\Pi}_{a}{}^{b}\,h^{2}{}_{be;c}\,h^{1\,dc;e}-3\,\tilde{P}^{1}{}_{a}{}^{b}\,h^{1}{}_{bc;e}\,h^{1\,dc;e}-\frac{3}{2}\tilde{\Pi}_{a}{}^{b}\,h^{2}{}_{bc;e}\,h^{1\,dc;e}-\frac{3}{2}\tilde{\Pi}^{db}\,h^{1}{}_{be;c}\,h^{2}{}_{a}{}^{c;e}+$$

$$\frac{3}{2}\tilde{\Pi}^{db}\, h^{1}{}_{bc;e}\, h^{2}{}_{a}{}^{c;e} + \frac{3}{2}\tilde{\Pi}_{a}{}^{b}\, h^{1}{}_{be;c}\, h^{2\,dc;e} - \frac{3}{2}\tilde{\Pi}_{a}{}^{b}\, h^{1}{}_{bc;e}\, h^{2\,dc;e} - \frac{3}{2} h^{1\,bc}\,\tilde{\Pi}^{de}\, h^{1}{}_{e}{}^{f}{}_{;a}\, h^{1}{}_{bc;f} +$$

$$\frac{3}{2} h^{1\,bc}\,\tilde{\Pi}_{a}{}^{e}\, h^{1}{}_{e}{}^{f;d}\, h^{1}{}_{bc;f} - \frac{3}{2} h^{1\,bc}\,\tilde{\Pi}^{de}\, h^{1}{}_{a}{}^{f}{}_{;e}\, h^{1}{}_{bc;f} + \frac{3}{2} h^{1\,bc}\,\tilde{\Pi}_{a}{}^{e}\, h^{1\,df}{}_{;e}\, h^{1}{}_{bc;f} +$$

$$3\, h^{1\,bc}\,\tilde{\Pi}^{de}\, h^{1}{}_{a}{}^{f}{}_{;b}\, h^{1}{}_{ce;f} - 3\, h^{1\,bc}\,\tilde{\Pi}_{a}{}^{e}\, h^{1\,df}{}_{;b}\, h^{1}{}_{ce;f} + 3\, h^{1\,bc}\,\tilde{\Pi}^{de}\, h^{1}{}_{be;a}\, h^{1}{}_{c}{}^{f}{}_{;f} - 3\, h^{1\,bc}\,\tilde{\Pi}^{de}\, h^{1}{}_{ae;b}\, h^{1}{}_{c}{}^{f}{}_{;f} +$$

$$3\, h^{1\,bc}\,\tilde{\Pi}_{a}{}^{e}\, h^{1\,d}{}_{e;b}\, h^{1}{}_{c}{}^{f}{}_{;f} - 3\, h^{1\,bc}\,\tilde{\Pi}_{a}{}^{e}\, h^{1}{}_{be}{}^{;d}\, h^{1}{}_{c}{}^{f}{}_{;f} + 3\, h^{1\,bc}\,\tilde{\Pi}^{de}\, h^{1}{}_{ab;e}\, h^{1}{}_{c}{}^{f}{}_{;f} - 3\, h^{1\,bc}\,\tilde{\Pi}_{a}{}^{e}\, h^{1\,d}{}_{b;e}\, h^{1}{}_{c}{}^{f}{}_{;f} -$$

$$3\, h^{1\,db}\,\tilde{\Pi}_{a}{}^{c}\, h^{1}{}_{c}{}^{e}{}_{;b}\, h^{1}{}_{e}{}^{f}{}_{;f} - 3\, h^{1\,db}\,\tilde{\Pi}_{a}{}^{c}\, h^{1}{}_{b}{}^{e}{}_{;c}\, h^{1}{}_{e}{}^{f}{}_{;f} + 3\, h^{1}{}_{a}{}^{b}\,\tilde{\Pi}_{b}{}^{c}\, h^{1\,de}{}_{;c}\, h^{1}{}_{e}{}^{f}{}_{;f} + 3\, h^{1}{}_{a}{}^{b}\,\tilde{\Pi}_{b}{}^{c}\, h^{1}{}_{c}{}^{e;d}\, h^{1}{}_{e}{}^{f}{}_{;f} +$$

$$3\, h^{1\,db}\,\tilde{\Pi}_{a}{}^{c}\, h^{1}{}_{bc}{}^{;e}\, h^{1}{}_{e}{}^{f}{}_{;f} - 3\, h^{1}{}_{a}{}^{b}\,\tilde{\Pi}_{b}{}^{c}\, h^{1\,d}{}_{c}{}^{;e}\, h^{1}{}_{e}{}^{f}{}_{;f} + 3\, h^{1\,ce}\, h^{1\,db}\,\tilde{\Pi}_{a}{}^{f}\, h^{1}{}_{ce;b;f} - 3\, h^{1}{}_{b}{}^{c}\, h^{1\,db}\,\tilde{\Pi}_{a}{}^{e}\, h^{1}{}_{e}{}^{f}{}_{;c;f} +$$

$$3\, h^{1}{}_{a}{}^{b}\, h^{1\,dc}\,\tilde{\Pi}_{b}{}^{e}\, h^{1}{}_{e}{}^{f}{}_{;c;f} + 3\, h^{1}{}_{b}{}^{e}\, h^{1\,bc}\,\tilde{\Pi}_{a}{}^{f}\, h^{1}{}_{ce}{}^{;d}{}_{;f} - 3\, h^{1}{}_{a}{}^{b}\, h^{1\,ce}\,\tilde{\Pi}_{b}{}^{f}\, h^{1}{}_{ce}{}^{;d}{}_{;f} - 3\, h^{1}{}_{b}{}^{c}\, h^{1\,db}\,\tilde{\Pi}_{a}{}^{e}\, h^{1}{}_{c}{}^{f}{}_{;e;f} +$$

$$3\, h^{1}{}_{a}{}^{b}\, h^{1\,dc}\,\tilde{\Pi}_{b}{}^{e}\, h^{1}{}_{c}{}^{f}{}_{;e;f} + 3\, h^{1}{}_{b}{}^{c}\, h^{1\,db}\,\tilde{\Pi}_{a}{}^{e}\, h^{1}{}_{ce}{}^{;f}{}_{;f} - 3\, h^{1}{}_{a}{}^{b}\, h^{1\,dc}\,\tilde{\Pi}_{b}{}^{e}\, h^{1}{}_{ce}{}^{;f}{}_{;f} + 3\, h^{1\,bc}\,\tilde{\Pi}^{de}\, h^{1}{}_{ef;c}\, h^{1}{}_{ab}{}^{;f} -$$

$$3\, h^{1\,bc}\,\tilde{\Pi}^{de}\, h^{1}{}_{ce;f}\, h^{1}{}_{ab}{}^{;f} - 3\, h^{1\,bc}\,\tilde{\Pi}^{de}\, h^{1}{}_{bf;c}\, h^{1}{}_{ae}{}^{;f} + \frac{3}{2} h^{1\,bc}\,\tilde{\Pi}^{de}\, h^{1}{}_{bc;f}\, h^{1}{}_{ae}{}^{;f} -$$

$$3\, h^{1\,db}\,\tilde{\Pi}_{a}{}^{c}\, h^{1}{}_{cf;e}\, h^{1}{}_{b}{}^{e;f} + 3\, h^{1\,db}\,\tilde{\Pi}_{a}{}^{c}\, h^{1}{}_{ce;f}\, h^{1}{}_{b}{}^{e;f} - 3\, h^{1\,bc}\,\tilde{\Pi}_{a}{}^{e}\, h^{1}{}_{ef;c}\, h^{1\,d}{}_{b}{}^{;f} + 3\, h^{1\,bc}\,\tilde{\Pi}_{a}{}^{e}\, h^{1}{}_{ce;f}\, h^{1\,d}{}_{b}{}^{;f} +$$

$$3\, h^{1\,bc}\,\tilde{\Pi}_{a}{}^{e}\, h^{1}{}_{bf;c}\, h^{1\,d}{}_{e}{}^{;f} - \frac{3}{2} h^{1\,bc}\,\tilde{\Pi}_{a}{}^{e}\, h^{1}{}_{bc;f}\, h^{1\,d}{}_{e}{}^{;f} + 3\, h^{1}{}_{a}{}^{b}\,\tilde{\Pi}_{b}{}^{c}\, h^{1}{}_{cf;e}\, h^{1\,de;f} - 3\, h^{1}{}_{a}{}^{b}\,\tilde{\Pi}_{b}{}^{c}\, h^{1}{}_{ce;f}\, h^{1\,de;f}$$

In the context of `xTensor'`, we can define scalar functions. This is a function of two arguments:

In[163]:=

```
DefScalarFunction[F, 2]
```

```
** DefScalarFunction: Defining scalar function F.
```

It must have scalar arguments. For example,

In[164]:=

```
F[RicciScalarCD[], T[a, b, c] RicciCD[-a, -b] v[-c]]
```

Out[164]=

$F[R,\ R_{ab}\, T^{abc}\, v_{c}]$

The command `Perturbation` knows how to apply the chain rule when acting on this kind of function at any order. It is essentially implemented through the *Mathematica* total derivative command `Dt`.

In[165]:=

```
Perturbation[%, 3]
```

Out[165]=

$$\begin{aligned}
&(6 \triangle[R_{ab}]\, \triangle[T^{abc}]\, \triangle[v_c] + 3 \triangle[v_c]\, \triangle^2[T^{abc}]\, R_{ab} + \\
&\quad 3 \triangle[T^{abc}]\, \triangle^2[v_c]\, R_{ab} + 3 \triangle[v_c]\, \triangle^2[R_{ab}]\, T^{abc} + 3 \triangle[R_{ab}]\, \triangle^2[v_c]\, T^{abc} + \triangle^3[v_c]\, R_{ab}\, T^{abc} + \\
&\quad 3 \triangle[T^{abc}]\, \triangle^2[R_{ab}]\, v_c + 3 \triangle[R_{ab}]\, \triangle^2[T^{abc}]\, v_c + \triangle^3[T^{abc}]\, R_{ab}\, v_c + \triangle^3[R_{ab}]\, T^{abc}\, v_c) \\
&\; F^{(0,1)}[R,\, R_{ab}\, T^{abc}\, v_c] + 3\, (\triangle[v_e]\, R_{ac}\, T^{ace} + \triangle[T^{ace}]\, R_{ac}\, v_e + \triangle[R_{ac}]\, T^{ace}\, v_e) \\
&\; (2 \triangle[T^{bdf}]\, \triangle[v_f]\, R_{bd} + 2 \triangle[R_{bd}]\, \triangle[v_f]\, T^{bdf} + \triangle^2[v_f]\, R_{bd}\, T^{bdf} + \\
&\quad 2 \triangle[R_{bd}]\, \triangle[T^{bdf}]\, v_f + \triangle^2[T^{bdf}]\, R_{bd}\, v_f + \triangle^2[R_{bd}]\, T^{bdf}\, v_f)\, F^{(0,2)}[R,\, R_{ab}\, T^{abc}\, v_c] + \\
&(\mathrm{Scalar}[\triangle[v_c]\, R_{ab}\, T^{abc}] + \mathrm{Scalar}[\triangle[T^{abc}]\, R_{ab}\, v_c] + \mathrm{Scalar}[\triangle[R_{ab}]\, T^{abc}\, v_c])^3 \\
&\; F^{(0,3)}[R,\, R_{ab}\, T^{abc}\, v_c] + \triangle^3[R]\, F^{(1,0)}[R,\, R_{ab}\, T^{abc}\, v_c] + \\
&3 \triangle^2[R]\, (\triangle[v_c]\, R_{ab}\, T^{abc} + \triangle[T^{abc}]\, R_{ab}\, v_c + \triangle[R_{ab}]\, T^{abc}\, v_c)\, F^{(1,1)}[R,\, R_{ab}\, T^{abc}\, v_c] + \\
&3 \triangle[R]\, (2 \triangle[T^{abc}]\, \triangle[v_c]\, R_{ab} + 2 \triangle[R_{ab}]\, \triangle[v_c]\, T^{abc} + \triangle^2[v_c]\, R_{ab}\, T^{abc} + \\
&\quad 2 \triangle[R_{ab}]\, \triangle[T^{abc}]\, v_c + \triangle^2[T^{abc}]\, R_{ab}\, v_c + \triangle^2[R_{ab}]\, T^{abc}\, v_c)\, F^{(1,1)}[R,\, R_{ab}\, T^{abc}\, v_c] + \\
&3 \triangle[R]\, (\mathrm{Scalar}[\triangle[v_c]\, R_{ab}\, T^{abc}] + \mathrm{Scalar}[\triangle[T^{abc}]\, R_{ab}\, v_c] + \mathrm{Scalar}[\triangle[R_{ab}]\, T^{abc}\, v_c])^2 \\
&\; F^{(1,2)}[R,\, R_{ab}\, T^{abc}\, v_c] + 3 \triangle[R]\, \triangle^2[R]\, F^{(2,0)}[R,\, R_{ab}\, T^{abc}\, v_c] + \\
&3\, \mathrm{Scalar}[\triangle[R]]^2\, (\triangle[v_c]\, R_{ab}\, T^{abc} + \triangle[T^{abc}]\, R_{ab}\, v_c + \triangle[R_{ab}]\, T^{abc}\, v_c)\, F^{(2,1)}[R,\, R_{ab}\, T^{abc}\, v_c] + \\
&\mathrm{Scalar}[\triangle[R]]^3\, F^{(3,0)}[R,\, R_{ab}\, T^{abc}\, v_c]
\end{aligned}$$

Again we find the head Scalar, to avoid problems with powers of scalars formed by contracted products of tensors.

■ 5. Gauge transformations

Through all this notebook, we have implicitly supposed a family of manifolds M(ϵ) equipped with metric g(ϵ), that depend on a dimensionless parameter ϵ. The member with parameter ϵ=0 will be called the background spacetime. The objective of perturbation theory is to compare perturbed objects, for instance a tensor T(ϵ), with their background counterparts , in this case T(0). The problem is that they are defined in different manifolds. Hence, we have to construct a mapping (the so−called gauge) ϕ(ϵ): M(ϵ) −−−−> M(0), whose pull−back ϕ*(ϵ) will take tensors defined in the tangent space of M(ϵ) to the tangent space of M(0). Then, the difference between perturbed and background tensors will define its perturbation. For instance, the full perturbation of the tensor T(0) is given by ϕ*(ϵ)(T(ϵ)) − T(0). This expression will be expanded in a power series of the parameter ϵ. Through this expansion, the different perturbative orders of the tensor T(0) will be defined. But, if we change the gauge ϕ to a different one ψ, the new perturbation of the tensor will change to ψ*(ϵ)(T(ϵ)) − T(0), which will affect to all the perturbative orders. The theory and formulas for the power expansion of changes of gauge was developed by Bruni et al. in reference Class. Quantum Grav. 14, 2585 (1997).

In the context of `xPert‘`, we have constructed a command named `GaugeChange[expr, ξ]` that performs the gauge transformation of the expression expr parameterized by the family of vector fields `ξ[LI[n],a]`, where `n` stands for the perturbative order.

We define the gauge generators. They must be a family of vector fields, one for each perturbative order. This is again taken into account by the label index `LI`.

In[166]:=

```
DefTensor[ξ[LI[n], a], M]
```

```
** DefTensor: Defining tensor ξ[LI[n], a].
```

The background objects do not depend on the perturbative gauge chosen, so they do not change under a gauge transformation.

In[167]:=
```
GaugeChange[
 g[-a, -b] RicciCD[c, d] + EinsteinCD[-a, c] EinsteinCD[-b, d] RicciScalarCD[], ξ]
```

Out[167]=
$$g_{ab}\, R^{cd} + G_a{}^c\, G_b{}^d\, R$$

The gauge transformation of the first−order metric perturbation is the following well−known formula.

In[168]:=
```
GaugeChange[h[LI[1], -a, -b], ξ]
```

Out[168]=
$$h^1{}_{ab} + \mathcal{L}_{\xi^1}\, g_{ab}$$

The first−order gauge transformation of any other object is similar:

In[169]:=
```
GaugeChange[τ[LI[1], -a, b, c], ξ]
```

Out[169]=
$$\tau^1{}_a{}^{bc} + \mathcal{L}_{\xi^1}\, T_a{}^{bc}$$

In[170]:=
```
% // ToCanonical
```

Out[170]=
$$\tau^1{}_a{}^{bc} + \mathcal{L}_{\xi^1}\, T_a{}^{bc}$$

At second order this formula is more complicated

In[171]:=
```
GaugeChange[h[LI[2], -a, -b], ξ]
```

Out[171]=
$$h^2{}_{ab} + 2\, (\mathcal{L}_{\xi^1}\, h^1{}_{ab}) + \mathcal{L}_{\xi^1}\, \mathcal{L}_{\xi^1}\, g_{ab} + \mathcal{L}_{\xi^2}\, g_{ab}$$

In[172]:=
```
% // ToCanonical
```

Out[172]=
$$h^2{}_{ab} + 2\, (\mathcal{L}_{\xi^1}\, h^1{}_{ab}) + \mathcal{L}_{\xi^1}\, \mathcal{L}_{\xi^1}\, g_{ab} + \mathcal{L}_{\xi^2}\, g_{ab}$$

Note that this previous formula is applied when the indices are in the natural position and that the metric, since it is a background object, commutes with the gauge change. This means that the following objects are the same:

In[173]:=
```
g[a, d] g[b, e] GaugeChange[τ[LI[3], -d, -e, c], ξ]
```

Out[173]=
$$g^{ad}\, (\tau^3{}_d{}^{bc} + 3\, (\mathcal{L}_{\xi^1}\, \tau^2{}_d{}^{bc}) + 3\, (\mathcal{L}_{\xi^1}\, \mathcal{L}_{\xi^1}\, \tau^1{}_d{}^{bc}) + \mathcal{L}_{\xi^1}\, \mathcal{L}_{\xi^1}\, \mathcal{L}_{\xi^1}\, T_d{}^{bc} + 3\, (\mathcal{L}_{\xi^1}\, \mathcal{L}_{\xi^2}\, T_d{}^{bc}) + 3\, (\mathcal{L}_{\xi^2}\, \tau^1{}_d{}^{bc}) + \mathcal{L}_{\xi^3}\, T_d{}^{bc})$$

In[174]:=
```
GaugeChange[τ[LI[3], a, b, c], ξ]
```

Out[174]=

$$g^{ad}\,(\tau^{3}{}_{d}{}^{bc} + 3\,(\mathcal{L}_{\xi^1}\,\tau^{2}{}_{d}{}^{bc}) + 3\,(\mathcal{L}_{\xi^1}\,\mathcal{L}_{\xi^1}\,\tau^{1}{}_{d}{}^{bc}) + \mathcal{L}_{\xi^1}\,\mathcal{L}_{\xi^1}\,\mathcal{L}_{\xi^1}\,T_{d}{}^{bc} + 3\,(\mathcal{L}_{\xi^1}\,\mathcal{L}_{\xi^2}\,T_{d}{}^{bc}) + 3\,(\mathcal{L}_{\xi^2}\,\tau^{1}{}_{d}{}^{bc}) + \mathcal{L}_{\xi^3}\,T_{d}{}^{bc})$$

In[175]:=
```
ToCanonical[%% - %, UseMetricOnVBundle → None]
```

0.132008 Second

Out[175]=
0

We can also do more difficult higher−order calculations. For instance,

In[176]:=
```
GaugeChange[τ[LI[2], a, b, c] h[LI[5], -a, -b], ξ]
```

Out[176]=

$$g^{ad}\,(\tau^{2}{}_{d}{}^{bc} + 2\,(\mathcal{L}_{\xi^1}\,\tau^{1}{}_{d}{}^{bc}) + \mathcal{L}_{\xi^1}\,\mathcal{L}_{\xi^1}\,T_{d}{}^{bc} + \mathcal{L}_{\xi^2}\,T_{d}{}^{bc})\,(h^{5}{}_{ab} + 5\,(\mathcal{L}_{\xi^1}\,h^{4}{}_{ab}) + 10\,(\mathcal{L}_{\xi^1}\,\mathcal{L}_{\xi^1}\,h^{3}{}_{ab}) + 10\,(\mathcal{L}_{\xi^1}\,\mathcal{L}_{\xi^1}\,\mathcal{L}_{\xi^1}\,h^{2}{}_{ab}) + 5\,(\mathcal{L}_{\xi^1}\,\mathcal{L}_{\xi^1}\,\mathcal{L}_{\xi^1}\,\mathcal{L}_{\xi^1}\,h^{1}{}_{ab}) + 30\,(\mathcal{L}_{\xi^1}\,\mathcal{L}_{\xi^1}\,\mathcal{L}_{\xi^2}\,h^{1}{}_{ab}) + \mathcal{L}_{\xi^1}\,\mathcal{L}_{\xi^1}\,\mathcal{L}_{\xi^1}\,\mathcal{L}_{\xi^1}\,\mathcal{L}_{\xi^1}\,g_{ab} + 10\,(\mathcal{L}_{\xi^1}\,\mathcal{L}_{\xi^1}\,\mathcal{L}_{\xi^1}\,\mathcal{L}_{\xi^2}\,g_{ab}) + 10\,(\mathcal{L}_{\xi^1}\,\mathcal{L}_{\xi^1}\,\mathcal{L}_{\xi^3}\,g_{ab}) + 30\,(\mathcal{L}_{\xi^1}\,\mathcal{L}_{\xi^2}\,h^{2}{}_{ab}) + 15\,(\mathcal{L}_{\xi^1}\,\mathcal{L}_{\xi^2}\,\mathcal{L}_{\xi^2}\,g_{ab}) + 20\,(\mathcal{L}_{\xi^1}\,\mathcal{L}_{\xi^3}\,h^{1}{}_{ab}) + 5\,(\mathcal{L}_{\xi^1}\,\mathcal{L}_{\xi^4}\,g_{ab}) + 10\,(\mathcal{L}_{\xi^2}\,h^{3}{}_{ab}) + 15\,(\mathcal{L}_{\xi^2}\,\mathcal{L}_{\xi^2}\,h^{1}{}_{ab}) + 10\,(\mathcal{L}_{\xi^2}\,\mathcal{L}_{\xi^3}\,g_{ab}) + 10\,(\mathcal{L}_{\xi^3}\,h^{2}{}_{ab}) + 5\,(\mathcal{L}_{\xi^4}\,h^{1}{}_{ab}) + \mathcal{L}_{\xi^5}\,g_{ab})$$

6. Background field perturbations

Making use of xPert`, we can also do background field perturbations. This approach is often used in quantum theory. There the perturbed metric is considered to be the background one plus a unique perturbation h. But when computing the inverse of the metric this unique perturbation will coupled to itself.

We can restrict our perturbation scheme to reproduce background field perturbations:

This can be implemented by forcing the vanishing of all perturbations of the metric of order greater than one, with the following rule:

In[177]:=
```
bgfield = h[LI[order_], __] :→ 0 /; order > 1;
```

Then, the perturbed metric would be

In[178]:=
```
Perturbed[g[-a, -b], 4] /. bgfield
```

Out[178]=

$$g_{ab} + \epsilon\,h^{1}{}_{ab}$$

And its inverse

In[179]:=

```
(Perturbed[g[b, c], 4] // ExpandPerturbation) /. bgfield // org
```

Out[179]=

$$g^{bc} - \epsilon\, h^{1\,bc} + \epsilon^2\, h^{1\,ba}\, h^{1\,c}{}_{a} - \epsilon^3\, h^{1}{}_{ad}\, h^{1\,ba}\, h^{1\,cd} + \epsilon^4\, h^{1}{}_{a}{}^{e}\, h^{1\,ba}\, h^{1\,cd}\, h^{1}{}_{de}$$

This is consistent because it leads to

In[180]:=

```
% %% // org
```

Out[180]=

$$\delta_{a}{}^{c} + \epsilon^5\, h^{1}{}_{a}{}^{b}\, h^{1}{}_{b}{}^{e}\, h^{1\,cd}\, h^{1}{}_{d}{}^{f}\, h^{1}{}_{ef}$$

We can compute the perturbation of any other geometric object in this context. For example the fourth−order perturbed Riemann will take the following form

In[181]:=

```
(Perturbed[RiemannCD[-a, -b, -c, d], 4] // ExpandPerturbation) /. bgfield // org
```

1.6401 Second

Out[181]=

$$\begin{aligned}
&R_{abc}{}^{d} + \epsilon \left(-\frac{h^{1}{}_{c}{}^{d}{}_{;b;a}}{2} - \frac{h^{1}{}_{b}{}^{d}{}_{;c;a}}{2} + \frac{h^{1}{}_{bc}{}^{;d}{}_{;a}}{2} + \frac{h^{1}{}_{c}{}^{d}{}_{;a;b}}{2} + \frac{h^{1}{}_{a}{}^{d}{}_{;c;b}}{2} - \frac{h^{1}{}_{ac}{}^{;d}{}_{;b}}{2} \right) + \\
&\epsilon^2 \Big(\frac{1}{2} h^{1\,de} h^{1}{}_{ce;b;a} + \frac{1}{2} h^{1\,de} h^{1}{}_{be;c;a} - \frac{1}{2} h^{1\,de} h^{1}{}_{bc;e;a} + \frac{1}{4} h^{1\,d}{}_{e;a} h^{1}{}_{c}{}^{e}{}_{;b} - \\
&\quad \frac{1}{4} h^{1}{}_{c}{}^{e}{}_{;a} h^{1\,d}{}_{e;b} - \frac{1}{2} h^{1\,de} h^{1}{}_{ce;a;b} - \frac{1}{2} h^{1\,de} h^{1}{}_{ae;c;b} + \frac{1}{2} h^{1\,de} h^{1}{}_{ac;e;b} - \frac{1}{4} h^{1\,d}{}_{e;b} h^{1}{}_{a}{}^{e}{}_{;c} + \\
&\quad \frac{1}{4} h^{1\,d}{}_{e;a} h^{1}{}_{b}{}^{e}{}_{;c} + \frac{1}{4} h^{1}{}_{ce;b} h^{1}{}_{a}{}^{e;d} + \frac{1}{4} h^{1}{}_{be;c} h^{1}{}_{a}{}^{e;d} - \frac{1}{4} h^{1}{}_{a}{}^{e}{}_{;c} h^{1}{}_{be}{}^{;d} - \frac{1}{4} h^{1}{}_{ce;a} h^{1}{}_{b}{}^{e;d} - \\
&\quad \frac{1}{4} h^{1}{}_{a}{}^{e;d} h^{1}{}_{bc;e} + \frac{1}{4} h^{1}{}_{a}{}^{e}{}_{;c} h^{1}{}_{b}{}^{d}{}_{;e} + \frac{1}{4} h^{1\,d}{}_{e;b} h^{1}{}_{ac}{}^{;e} + \frac{1}{4} h^{1}{}_{be}{}^{;d} h^{1}{}_{ac}{}^{;e} - \frac{1}{4} h^{1}{}_{b}{}^{d}{}_{;e} h^{1}{}_{ac}{}^{;e} - \\
&\quad \frac{1}{4} h^{1}{}_{ce;b} h^{1}{}_{a}{}^{d;e} - \frac{1}{4} h^{1}{}_{be;c} h^{1}{}_{a}{}^{d;e} + \frac{1}{4} h^{1}{}_{bc;e} h^{1}{}_{a}{}^{d;e} - \frac{1}{4} h^{1\,d}{}_{e;a} h^{1}{}_{bc}{}^{;e} + \frac{1}{4} h^{1}{}_{ce;a} h^{1}{}_{b}{}^{d;e} \Big) + \\
&\epsilon^3 \Big(-\frac{1}{2} h^{1\,de} h^{1}{}_{e}{}^{f} h^{1}{}_{cf;b;a} - \frac{1}{2} h^{1\,de} h^{1}{}_{e}{}^{f} h^{1}{}_{bf;c;a} + \frac{1}{2} h^{1\,de} h^{1}{}_{e}{}^{f} h^{1}{}_{bc;f;a} - \frac{1}{4} h^{1\,ef} h^{1\,d}{}_{f;a} h^{1}{}_{ce;b} - \\
&\quad \frac{1}{4} h^{1\,de} h^{1}{}_{ef;a} h^{1}{}_{c}{}^{f}{}_{;b} + \frac{1}{4} h^{1\,ef} h^{1}{}_{ce;a} h^{1\,d}{}_{f;b} + \frac{1}{4} h^{1\,de} h^{1}{}_{c}{}^{f}{}_{;a} h^{1}{}_{ef;b} + \\
&\quad \frac{1}{2} h^{1\,de} h^{1}{}_{e}{}^{f} h^{1}{}_{cf;a;b} + \frac{1}{2} h^{1\,de} h^{1}{}_{e}{}^{f} h^{1}{}_{af;c;b} - \frac{1}{2} h^{1\,de} h^{1}{}_{e}{}^{f} h^{1}{}_{ac;f;b} + \frac{1}{4} h^{1\,ef} h^{1\,d}{}_{f;b} h^{1}{}_{ae;c} + \\
&\quad \frac{1}{4} h^{1\,de} h^{1}{}_{ef;b} h^{1}{}_{a}{}^{f}{}_{;c} - \frac{1}{4} h^{1\,ef} h^{1\,d}{}_{f;a} h^{1}{}_{be;c} - \frac{1}{4} h^{1\,de} h^{1}{}_{ef;a} h^{1}{}_{b}{}^{f}{}_{;c} - \frac{1}{4} h^{1\,ef} h^{1}{}_{cf;b} h^{1}{}_{ae}{}^{;d} - \\
&\quad \frac{1}{4} h^{1\,ef} h^{1}{}_{bf;c} h^{1}{}_{ae}{}^{;d} + \frac{1}{4} h^{1\,ef} h^{1}{}_{cf;a} h^{1}{}_{be}{}^{;d} + \frac{1}{4} h^{1\,ef} h^{1}{}_{ae;c} h^{1}{}_{bf}{}^{;d} - \frac{1}{4} h^{1\,ef} h^{1\,d}{}_{f;b} h^{1}{}_{ac;e} - \\
&\quad \frac{1}{4} h^{1\,ef} h^{1}{}_{bf}{}^{;d} h^{1}{}_{ac;e} + \frac{1}{4} h^{1\,ef} h^{1}{}_{cf;b} h^{1}{}_{a}{}^{d}{}_{;e} + \frac{1}{4} h^{1\,ef} h^{1}{}_{bf;c} h^{1}{}_{a}{}^{d}{}_{;e} - \frac{1}{4} h^{1\,de} h^{1}{}_{c}{}^{f}{}_{;b} h^{1}{}_{af;e} - \\
&\quad \frac{1}{4} h^{1\,de} h^{1}{}_{b}{}^{f}{}_{;c} h^{1}{}_{af;e} + \frac{1}{4} h^{1\,ef} h^{1\,d}{}_{f;a} h^{1}{}_{bc;e} - \frac{1}{4} h^{1\,ef} h^{1}{}_{cf;a} h^{1}{}_{b}{}^{d}{}_{;e} + \frac{1}{4} h^{1\,de} h^{1}{}_{c}{}^{f}{}_{;a} h^{1}{}_{bf;e} + \\
&\quad \frac{1}{4} h^{1\,de} h^{1}{}_{a}{}^{f}{}_{;c} h^{1}{}_{bf;e} + \frac{1}{4} h^{1\,de} h^{1}{}_{c}{}^{f}{}_{;b} h^{1}{}_{ae;f} + \frac{1}{4} h^{1\,de} h^{1}{}_{b}{}^{f}{}_{;c} h^{1}{}_{ae;f} + \frac{1}{4} h^{1\,ef} h^{1}{}_{ae}{}^{;d} h^{1}{}_{bc;f} - \\
&\quad \frac{1}{4} h^{1\,ef} h^{1}{}_{a}{}^{d}{}_{;e} h^{1}{}_{bc;f} - \frac{1}{4} h^{1\,ef} h^{1}{}_{ae;c} h^{1}{}_{b}{}^{d}{}_{;f} + \frac{1}{4} h^{1\,ef} h^{1}{}_{ac;e} h^{1}{}_{b}{}^{d}{}_{;f} - \frac{1}{4} h^{1\,de} h^{1}{}_{c}{}^{f}{}_{;a} h^{1}{}_{be;f} - \\
&\quad \frac{1}{4} h^{1\,de} h^{1}{}_{a}{}^{f}{}_{;c} h^{1}{}_{be;f} - \frac{1}{4} h^{1\,de} h^{1}{}_{ef;b} h^{1}{}_{ac}{}^{;f} - \frac{1}{4} h^{1\,de} h^{1}{}_{bf;e} h^{1}{}_{ac}{}^{;f} + \frac{1}{4} h^{1\,de} h^{1}{}_{be;f} h^{1}{}_{ac}{}^{;f} +
\end{aligned}$$

$$\frac{1}{4} h^{1\,de} h^1{}_{ef;a} h^1{}_{bc}{}^{;f} + \frac{1}{4} h^{1\,de} h^1{}_{af;e} h^1{}_{bc}{}^{;f} - \frac{1}{4} h^{1\,de} h^1{}_{ae;f} h^1{}_{bc}{}^{;f} \Big) +$$

$$\epsilon^4 \Big(\frac{1}{2} h^{1\,de} h^1{}_e{}^f h^1{}_f{}^{f1} h^1{}_{cf1;b;a} + \frac{1}{2} h^{1\,de} h^1{}_e{}^f h^1{}_f{}^{f1} h^1{}_{bf1;c;a} - \frac{1}{2} h^{1\,de} h^1{}_e{}^f h^1{}_f{}^{f1} h^1{}_{bc;f1;a} +$$

$$\frac{1}{4} h^1{}_e{}^{f1} h^{1\,ef} h^{1\,d}{}_{f1;a} h^1{}_{cf;b} + \frac{1}{4} h^{1\,de} h^{1\,ff1} h^1{}_{ef1;a} h^1{}_{cf;b} + \frac{1}{4} h^{1\,de} h^1{}_e{}^f h^1{}_{ff1;a} h^1{}_c{}^{f1}{}_{;b} -$$

$$\frac{1}{4} h^1{}_e{}^{f1} h^{1\,ef} h^1{}_{cf;a} h^{1\,d}{}_{f1;b} - \frac{1}{4} h^{1\,de} h^{1\,ff1} h^1{}_{cf;a} h^1{}_{ef1;b} - \frac{1}{4} h^{1\,de} h^1{}_e{}^f h^1{}_c{}^{f1}{}_{;a} h^1{}_{ff1;b} -$$

$$\frac{1}{2} h^{1\,de} h^1{}_e{}^f h^1{}_f{}^{f1} h^1{}_{cf1;a;b} - \frac{1}{2} h^{1\,de} h^1{}_e{}^f h^1{}_f{}^{f1} h^1{}_{af1;c;b} + \frac{1}{2} h^{1\,de} h^1{}_e{}^f h^1{}_f{}^{f1} h^1{}_{ac;f1;b} -$$

$$\frac{1}{4} h^1{}_e{}^{f1} h^{1\,ef} h^{1\,d}{}_{f1;b} h^1{}_{af;c} - \frac{1}{4} h^{1\,de} h^{1\,ff1} h^1{}_{ef1;b} h^1{}_{af;c} - \frac{1}{4} h^{1\,de} h^1{}_e{}^f h^1{}_{ff1;b} h^1{}_a{}^{f1}{}_{;c} +$$

$$\frac{1}{4} h^1{}_e{}^{f1} h^{1\,ef} h^{1\,d}{}_{f1;a} h^1{}_{bf;c} + \frac{1}{4} h^{1\,de} h^{1\,ff1} h^1{}_{ef1;a} h^1{}_{bf;c} + \frac{1}{4} h^{1\,de} h^1{}_e{}^f h^1{}_{ff1;a} h^1{}_b{}^{f1}{}_{;c} +$$

$$\frac{1}{4} h^1{}_e{}^{f1} h^{1\,ef} h^1{}_{cf1;b} h^1{}_{af}{}^{;d} + \frac{1}{4} h^1{}_e{}^{f1} h^{1\,ef} h^1{}_{bf1;c} h^1{}_{af}{}^{;d} - \frac{1}{4} h^1{}_e{}^{f1} h^{1\,ef} h^1{}_{cf1;a} h^1{}_{bf}{}^{;d} -$$

$$\frac{1}{4} h^1{}_e{}^{f1} h^{1\,ef} h^1{}_{af;c} h^1{}_{bf1}{}^{;d} + \frac{1}{4} h^{1\,de} h^{1\,ff1} h^1{}_{cf;b} h^1{}_{af1;e} + \frac{1}{4} h^{1\,de} h^{1\,ff1} h^1{}_{bf;c} h^1{}_{af1;e} -$$

$$\frac{1}{4} h^{1\,de} h^{1\,ff1} h^1{}_{cf;a} h^1{}_{bf1;e} - \frac{1}{4} h^{1\,de} h^{1\,ff1} h^1{}_{af;c} h^1{}_{bf1;e} + \frac{1}{4} h^1{}_e{}^{f1} h^{1\,ef} h^{1\,d}{}_{f1;b} h^1{}_{ac;f} +$$

$$\frac{1}{4} h^{1\,de} h^{1\,ff1} h^1{}_{ef1;b} h^1{}_{ac;f} + \frac{1}{4} h^1{}_e{}^{f1} h^{1\,ef} h^1{}_{bf1}{}^{;d} h^1{}_{ac;f} + \frac{1}{4} h^{1\,de} h^{1\,ff1} h^1{}_{bf1;e} h^1{}_{ac;f} -$$

$$\frac{1}{4} h^1{}_e{}^{f1} h^{1\,ef} h^1{}_{cf1;b} h^1{}_a{}^d{}_{;f} - \frac{1}{4} h^1{}_e{}^{f1} h^{1\,ef} h^1{}_{bf1;c} h^1{}_a{}^d{}_{;f} + \frac{1}{4} h^{1\,de} h^1{}_e{}^f h^1{}_c{}^{f1}{}_{;b} h^1{}_{af1;f} +$$

$$\frac{1}{4} h^{1\,de} h^1{}_e{}^f h^1{}_b{}^{f1}{}_{;c} h^1{}_{af1;f} - \frac{1}{4} h^1{}_e{}^{f1} h^{1\,ef} h^{1\,d}{}_{f1;a} h^1{}_{bc;f} - \frac{1}{4} h^{1\,de} h^{1\,ff1} h^1{}_{ef1;a} h^1{}_{bc;f} -$$

$$\frac{1}{4} h^{1\,de} h^{1\,ff1} h^1{}_{af1;e} h^1{}_{bc;f} + \frac{1}{4} h^1{}_e{}^{f1} h^{1\,ef} h^1{}_{cf1;a} h^1{}_b{}^d{}_{;f} - \frac{1}{4} h^{1\,de} h^1{}_e{}^f h^1{}_c{}^{f1}{}_{;a} h^1{}_{bf1;f} -$$

$$\frac{1}{4} h^{1\,de} h^1{}_e{}^f h^1{}_a{}^{f1}{}_{;c} h^1{}_{bf1;f} - \frac{1}{4} h^{1\,de} h^{1\,ff1} h^1{}_{cf;b} h^1{}_{ae;f1} - \frac{1}{4} h^{1\,de} h^{1\,ff1} h^1{}_{bf;c} h^1{}_{ae;f1} +$$

$$\frac{1}{4} h^{1\,de} h^{1\,ff1} h^1{}_{bc;f} h^1{}_{ae;f1} - \frac{1}{4} h^{1\,de} h^1{}_e{}^f h^1{}_c{}^{f1}{}_{;b} h^1{}_{af;f1} - \frac{1}{4} h^{1\,de} h^1{}_e{}^f h^1{}_b{}^{f1}{}_{;c} h^1{}_{af;f1} -$$

$$\frac{1}{4} h^1{}_e{}^{f1} h^{1\,ef} h^1{}_{af}{}^{;d} h^1{}_{bc;f1} + \frac{1}{4} h^1{}_e{}^{f1} h^{1\,ef} h^1{}_a{}^d{}_{;f} h^1{}_{bc;f1} + \frac{1}{4} h^1{}_e{}^{f1} h^{1\,ef} h^1{}_{af;c} h^1{}_b{}^d{}_{;f1} -$$

$$\frac{1}{4} h^1{}_e{}^{f1} h^{1\,ef} h^1{}_{ac;f} h^1{}_b{}^d{}_{;f1} + \frac{1}{4} h^{1\,de} h^{1\,ff1} h^1{}_{cf;a} h^1{}_{be;f1} + \frac{1}{4} h^{1\,de} h^{1\,ff1} h^1{}_{af;c} h^1{}_{be;f1} -$$

$$\frac{1}{4} h^{1\,de} h^{1\,ff1} h^1{}_{ac;f} h^1{}_{be;f1} + \frac{1}{4} h^{1\,de} h^1{}_e{}^f h^1{}_c{}^{f1}{}_{;a} h^1{}_{bf;f1} + \frac{1}{4} h^{1\,de} h^1{}_e{}^f h^1{}_a{}^{f1}{}_{;c} h^1{}_{bf;f1} +$$

$$\frac{1}{4} h^{1\,de} h^1{}_e{}^f h^1{}_{ff1;b} h^1{}_{ac}{}^{;f1} + \frac{1}{4} h^{1\,de} h^1{}_e{}^f h^1{}_{bf1;f} h^1{}_{ac}{}^{;f1} - \frac{1}{4} h^{1\,de} h^1{}_e{}^f h^1{}_{bf;f1} h^1{}_{ac}{}^{;f1} -$$

$$\frac{1}{4} h^{1\,de} h^1{}_e{}^f h^1{}_{ff1;a} h^1{}_{bc}{}^{;f1} - \frac{1}{4} h^{1\,de} h^1{}_e{}^f h^1{}_{af1;f} h^1{}_{bc}{}^{;f1} + \frac{1}{4} h^{1\,de} h^1{}_e{}^f h^1{}_{af;f1} h^1{}_{bc}{}^{;f1} \Big)$$

■ 7.Timings

This section performs a number of experiments to analyze the efficiency of the system. All timings have been performed with a Pentium IV 3GHz Linux box with 2Gbytes of RAM running *Mathematica* 5.2.

We load some `Graphics‘` packages

```
In[182]:=
      << Graphics‘Graphics‘
```

```
In[183]:=
      << Graphics‘MultipleListPlot‘
```

6.1. Timings with perturbations of the Riemann tensor

We compute the perturbations of the Riemann tensor in terms of metric perturbations up to tenth order. We measure the time `ExpandPerturbation` takes to calculate each order and make a plot of these timings versus the perturbative order. We also consider the the graph of the number of terms versus perturbative order.

Perturbations of the Riemann tensor up to tenth order. Save the results of timing and the perturbative expression itself in pairs (ti, ri).

```
In[184]:=
    {t1, r1} = AbsoluteTiming[
      ExpandPerturbation[Perturbation[RiemannCD[-a, -b, -c, d], 1]] // ToCanonical];

In[185]:=
    {t2, r2} = AbsoluteTiming[
      ExpandPerturbation[Perturbation[RiemannCD[-a, -b, -c, d], 2]] // ToCanonical];

      0.244015 Second

In[186]:=
    {t3, r3} = AbsoluteTiming[
      ExpandPerturbation[Perturbation[RiemannCD[-a, -b, -c, d], 3]] // ToCanonical];

      0.888055 Second

In[187]:=
    {t4, r4} = AbsoluteTiming[
      ExpandPerturbation[Perturbation[RiemannCD[-a, -b, -c, d], 4]] // ToCanonical];

      2.77217 Second

In[188]:=
    {t5, r5} = AbsoluteTiming[
      ExpandPerturbation[Perturbation[RiemannCD[-a, -b, -c, d], 5]] // ToCanonical];

      7.9685 Second

In[189]:=
    {t6, r6} = AbsoluteTiming[
      ExpandPerturbation[Perturbation[RiemannCD[-a, -b, -c, d], 6]] // ToCanonical];

      21.5853 Second

In[190]:=
    {t7, r7} = AbsoluteTiming[
      ExpandPerturbation[Perturbation[RiemannCD[-a, -b, -c, d], 7]] // ToCanonical];

      60.7638 Second

In[191]:=
    {t8, r8} = AbsoluteTiming[
      ExpandPerturbation[Perturbation[RiemannCD[-a, -b, -c, d], 8]] // ToCanonical];

      159.066 Second

In[192]:=
    {t9, r9} = AbsoluteTiming[
      ExpandPerturbation[Perturbation[RiemannCD[-a, -b, -c, d], 9]] // ToCanonical];

      408.962 Second
```

In[193]:=
```
{t10, r10} = AbsoluteTiming[
   ExpandPerturbation[Perturbation[RiemannCD[-a, -b, -c, d], 10]] // ToCanonical];
```

1134.67 Second

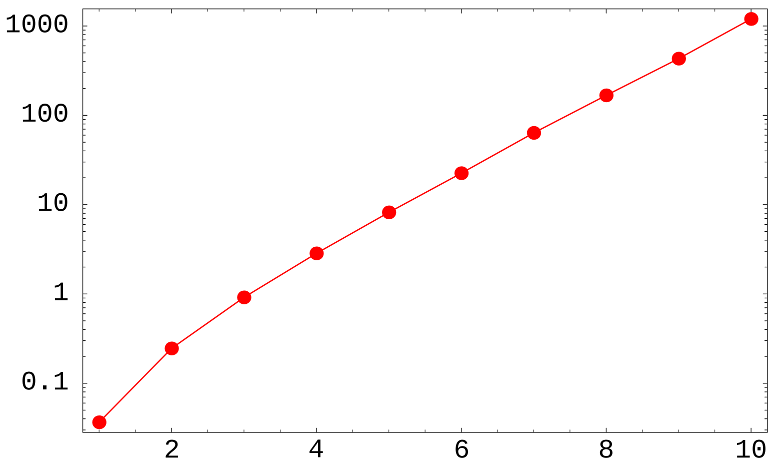
The following is the list of the times measured above

In[194]:=
```
times = {t1, t2, t3, t4, t5, t6, t7, t8, t9, t10} /. Second → 1
```

Out[194]=
```
{0.036555, 0.245376, 0.914745, 2.846096, 8.191186,
 22.481673, 63.588681, 167.305122, 431.159441, 1199.161759}
```

We plot then in a logarithmic plot

In[195]:=
```
timesplot = Show[LogListPlot[times, PlotStyle → Hue[0],
   Frame → True, Axes → False, PlotJoined → True, DisplayFunction → Identity],
  LogListPlot[times, PlotStyle → {Hue[0], PointSize[0.02]}, Frame → True,
   Axes → False, SymbolShape → PlotSymbol[Star, Filled → False],
   DisplayFunction → Identity], DisplayFunction → $DisplayFunction]
```



Out[195]=
- Graphics -

This is the list of the lengths of each perturbative expression

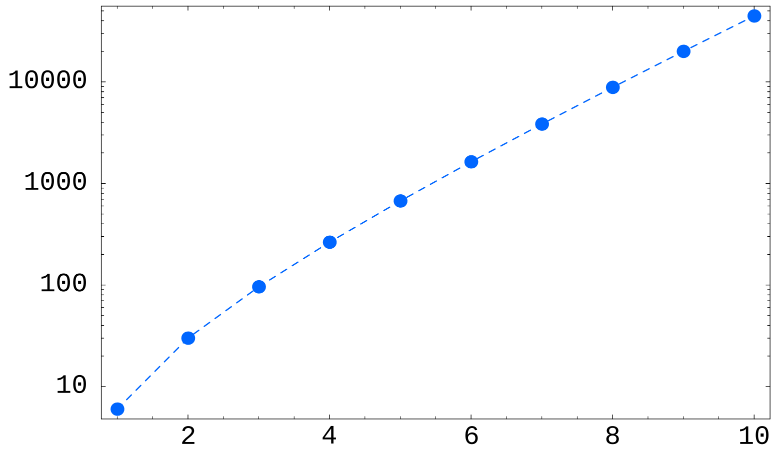
In[196]:=
```
lengths = Length /@ Expand[{r1, r2, r3, r4, r5, r6, r7, r8, r9, r10}]
```

1.38809 Second

Out[196]=
```
{6, 30, 96, 264, 672, 1632, 3840, 8832, 19968, 44544}
```

We also plot then in a logaritmic plot

```
In[197]:=
     lengthsplot = Show[LogListPlot[lengths, PlotStyle → {Hue[0.6], Dashing[{0.01, 0.01}]},
         Frame → True, Axes → False, PlotJoined → True, DisplayFunction → Identity],
        LogListPlot[lengths, PlotStyle → {Hue[0.6], PointSize[0.02]}, Frame → True,
         Axes → False, DisplayFunction → Identity], DisplayFunction → $DisplayFunction]
```



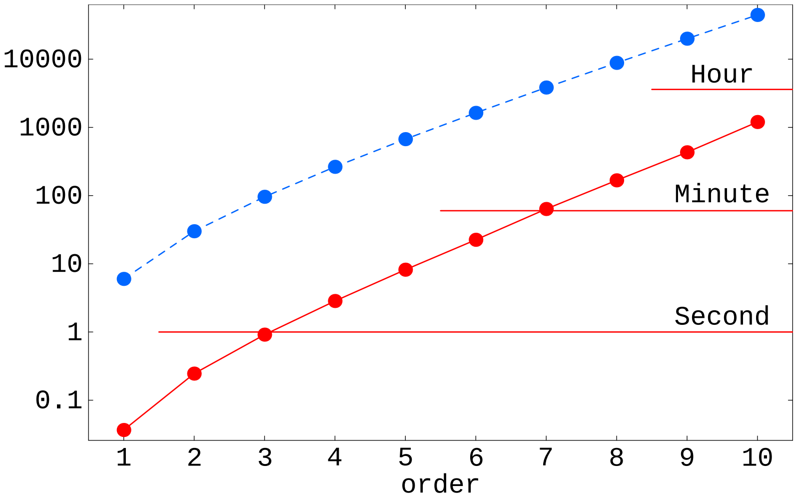

```
Out[197]=
     - Graphics -
```

Prepare ticks distribution:

```
In[198]:=
     tickat[n_] := {n, n, {0.00625, 0}, {GrayLevel[0], AbsoluteThickness[0.25]}};
     ticks1 = tickat /@ Range[10];
     logtickat[n_] := {Log[10, n], If[n < 1, N[n], n],
         {0.00625, 0}, {GrayLevel[0], AbsoluteThickness[0.25]}};
     ticks2 = logtickat /@ (10^Range[-4, 7]);
     tickat[n_] := {n, "", {0.00625, 0}, {GrayLevel[0], AbsoluteThickness[0.25]}};
     ticks3 = tickat /@ Range[10];
     logtickat[n_] :=
        {Log[10, n], "", {0.00625, 0}, {GrayLevel[0], AbsoluteThickness[0.25]}};
     ticks4 = logtickat /@ (10^Range[-4, 7]);
     ticks = {ticks1, ticks2, ticks3, ticks4};
```

and show both graphics together

```
In[207]:=
      plot = Show[timesplot,
        Graphics[{Hue[0], Line[{{1.5, 0}, {10.5, 0}}]}],
        Graphics[{Hue[0], Line[{{5.5, Log[10, 60]}, {10.5, Log[10, 60]}}]}],
        Graphics[{Hue[0], Line[{{8.5, Log[10, 3600]}, {10.5, Log[10, 3600]}}]}],
        lengthsplot, Graphics[Text["Second", {9.5, 0.2}]],
        Graphics[Text["Minute", {9.5, 2}]], Graphics[Text["Hour", {9.5, 3.75}]],
        PlotRange → {{0.5, 10.5}, All}, ImageSize → 300,
        FrameLabel → {order, None}, FrameTicks → ticks]
```




```
Out[207]=
      - Graphics -
```

6.2. Timings with the Leibnitz rule

We now apply `Perturbation` on products of different numbers of objects. We then graphically represent the time it takes for the expression to be expanded versus the perturbative order and the number of factors in the product. The *n*−th order perturbation of the Leibnitz rule is implemented in the context of `xPert‘` through a closed formula. We will see that this is faster than the chain rule, that is implemented essentialy via the *Mathematica* command `Dt`.

Driver to average a number of experiments. For timings below a second the computation is repeated a number of times:

```
In[208]:=
      average[lists_List] := Plus @@ lists / Length[lists];
      threshold = 1;
      experiment[number_Integer, order_Integer] :=
        Module[{timing = experiment1[number, order], repeat},
         If[timing[[1, 1]] < threshold,
           repeat = Max[1, Ceiling[threshold / timing[[1, 1]]]];
           average@Table[experiment1[number, order], {repeat}],
           timing] /. Second → 1
        ];
```

Generate products of n>1 arguments:

```
In[211]:=
      Table[prod[n] = Times @@ Table[Unique[x], {n}], {n, 1, 10}];
```

and measure the time Perturbation takes to act on them, and the number of terms produced:

In[212]:=
```
experiment1[number_, 1] :=
  AbsoluteTiming[Length@Expand@Perturbation[prod[number]]];
experiment1[number_, order_] :=
  AbsoluteTiming[Length@Expand@Perturbation[prod[number], order]];
```

In[214]:=
```
Llist[1] = experiment[#, 1] & /@ Range[2, 10]
```

8.67654 Second

Out[214]=
```
{{0.000211, 2}, {0.000332, 3}, {0.000558, 4}, {0.000827, 5},
 {0.001268, 6}, {0.001962, 7}, {0.003622, 8}, {0.005935, 9}, {0.010834, 10}}
```

In[215]:=
```
Llist[2] = experiment[#, 2] & /@ Range[2, 10]
```

8.74455 Second

Out[215]=
```
{{0.000267, 3}, {0.000582, 6}, {0.001080, 10}, {0.001836, 15},
 {0.002867, 21}, {0.004534, 28}, {0.007061, 36}, {0.011499, 45}, {0.017347, 55}}
```

In[216]:=
```
Llist[3] = experiment[#, 3] & /@ Range[2, 10]
```

8.20451 Second

Out[216]=
```
{{0.000333, 4}, {0.000897, 10}, {0.001975, 20}, {0.003841, 35}, {0.006783, 56},
 {0.011084, 84}, {0.018257, 120}, {0.028401, 165}, {0.045301, 220}}
```

In[217]:=
```
Llist[4] = experiment[#, 4] & /@ Range[2, 10]
```

8.75255 Second

Out[217]=
```
{{0.000399, 5}, {0.001293, 15}, {0.003245, 35}, {0.007178, 70}, {0.015798, 126},
 {0.026700, 210}, {0.046359, 330}, {0.074470, 495}, {0.117205, 715}}
```

In[218]:=
```
Llist[5] = experiment[#, 5] & /@ Range[2, 10]
```

9.81261 Second

Out[218]=
```
{{0.000461, 6}, {0.001817, 21}, {0.005189, 56}, {0.013916, 126}, {0.030750, 252},
 {0.060912, 462}, {0.127107, 792}, {0.219005, 1287}, {0.352478, 2002}}
```

In[219]:=
```
Llist[6] = experiment[#, 6] & /@ Range[2, 10]
```

11.5687 Second

Out[219]=
```
{{0.000597, 7}, {0.002702, 28}, {0.008821, 84}, {0.025633, 210}, {0.057140, 462},
 {0.136063, 924}, {0.260038, 1716}, {0.510616, 3003}, {0.951655, 5005}}
```

In[220]:=
```
Llist[7] = experiment[#, 7] & /@ Range[2, 10]
```

11.6247 Second

Out[220]=
```
{{0.000694, 8}, {0.003526, 36}, {0.012212, 120}, {0.038457, 330}, {0.102152, 792},
 {0.246908, 1716}, {0.551533, 3432}, {1.080255, 6435}, {2.041409, 11440}}
```

In[221]:=
```
Llist[8] = experiment[#, 8] & /@ Range[2, 10]
```

15.277 Second

Out[221]=
```
{{0.000898, 9}, {0.004340, 45}, {0.017768, 165}, {0.052049, 495}, {0.156999, 1287},
 {0.401607, 3003}, {0.906162, 6435}, {1.957846, 12870}, {3.988170, 24310}}
```

In[222]:=
```
Llist[9] = experiment[#, 9] & /@ Range[2, 10]
```

22.4534 Second

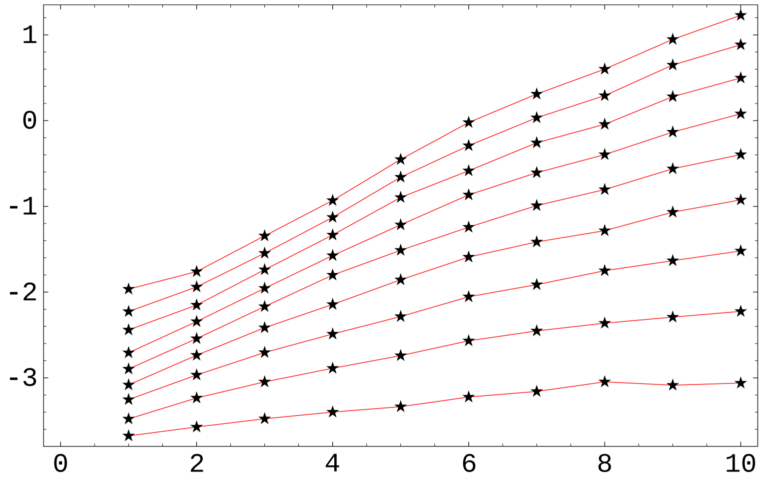
Out[222]=
```
{{0.000821, 10}, {0.005109, 55}, {0.023245, 220}, {0.085614, 715}, {0.274254, 2002},
 {0.735193, 5005}, {1.903072, 11440}, {4.452425, 24310}, {8.864111, 48620}}
```

In[223]:=
```
Llist[10] = experiment[#, 10] & /@ Range[2, 10]
```

34.3101 Second

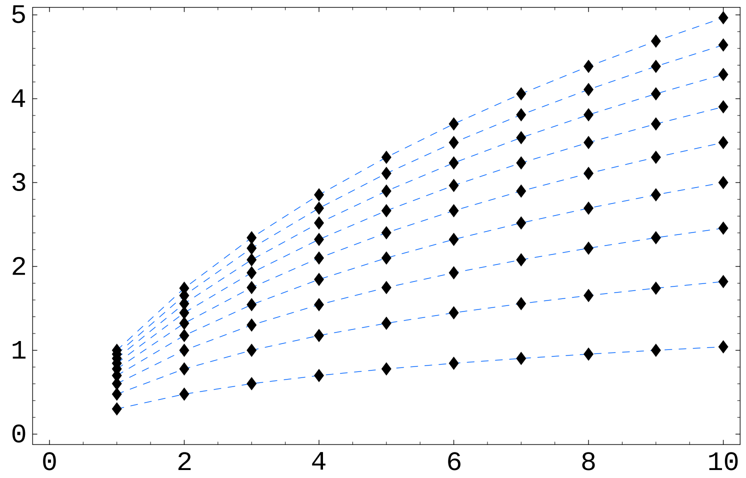
Out[223]=
```
{{0.000869, 11}, {0.005958, 66}, {0.030211, 286}, {0.118979, 1001}, {0.401679, 3003},
 {1.200730, 8008}, {3.144754, 19448}, {7.678046, 43758}, {16.930981, 92378}}
```

In[224]:=
```
Ltimingsplot = MultipleListPlot[
  Transpose@Log[10, Map[First, Array[Llist, {10}], {2}]], PlotJoined → True,
  Axes → False, Frame → True, SymbolShape → PlotSymbol[Star], PlotStyle → Hue[0]]
```



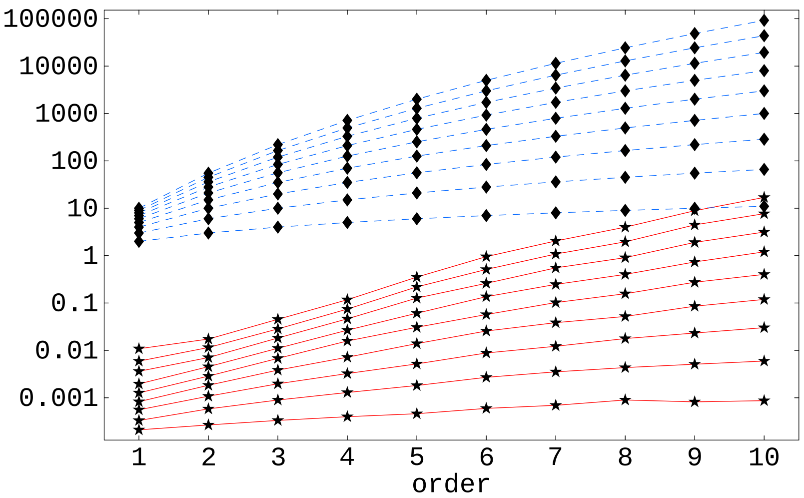
Out[224]=
```
- Graphics -
```

```
In[225]:=
      Ltermsplot = MultipleListPlot[Transpose@Log[10, Map[Last, Array[Llist, {10}], {2}]],
        PlotJoined → True, Axes → False, Frame → True, SymbolShape → PlotSymbol[Diamond],
        PlotStyle → {{Hue[0.6], Dashing[{0.01, 0.01}]}}]
```



```
Out[225]=
      - Graphics -

In[226]:=
      Lplot = Show[Ltimingsplot, Ltermsplot, FrameLabel → {order, None},
        FrameTicks → ticks, PlotRange → {{0.5, 10.5}, All}, ImageSize → 300]
```




```
Out[226]=
      - Graphics -
```

6.2. Timings with the chain rule

In this section we measure the time it takes to the command `Perturbation` to act on a scalar function of up to 10 arguments through the chain rule. We analize it up to the fifth perturbative order. In the context of `xPert`‘, we have implemented the nth order perturbation of the chain rule trough the *Mathematica* command `Dt`.

Generate functions of n arguments and perturb them:

In[227]:=
```
Table[func[n] = F @@ Table[Unique[x], {n}], {n, 1, 10}];
```

In[228]:=
```
experiment1[number_, 1] :=
  AbsoluteTiming[Length@Expand@Perturbation[func[number]]];
experiment1[number_, order_] :=
  AbsoluteTiming[Length@Expand@Perturbation[func[number], order]];
```

In[230]:=
```
Flist[1] = experiment[#, 1] & /@ Range[10]
```

```
9.70461 Second
```

Out[230]=
```
{{0.000401, 2}, {0.000665, 2}, {0.000944, 3}, {0.001162, 4}, {0.001436, 5},
 {0.001694, 6}, {0.001974, 7}, {0.002245, 8}, {0.002539, 9}, {0.002852, 10}}
```

The first case is a product of two objects and hence its length must be corrected:

In[231]:=
```
Flist[1] = ReplacePart[Flist[1], 1, {1, 2}]
```

Out[231]=
```
{{0.000401, 1}, {0.000665, 2}, {0.000944, 3}, {0.001162, 4}, {0.001436, 5},
 {0.001694, 6}, {0.001974, 7}, {0.002245, 8}, {0.002539, 9}, {0.002852, 10}}
```

In[232]:=
```
Flist[2] = experiment[#, 2] & /@ Range[10]
```

```
9.07657 Second
```

Out[232]=
```
{{0.000587, 2}, {0.001233, 5}, {0.001967, 9}, {0.002857, 14}, {0.003935, 20},
 {0.005273, 27}, {0.006778, 35}, {0.008543, 44}, {0.010484, 54}, {0.012736, 65}}
```

In[233]:=
```
Flist[3] = experiment[#, 3] & /@ Range[10]
```

```
9.97662 Second
```

Out[233]=
```
{{0.000870, 3}, {0.002557, 10}, {0.005340, 22}, {0.009621, 40}, {0.015884, 65},
 {0.024725, 98}, {0.036190, 140}, {0.051157, 192}, {0.071447, 255}, {0.096454, 330}}
```

In[234]:=
```
Flist[4] = experiment[#, 4] & /@ Range[10]
```

```
13.8609 Second
```

Out[234]=
```
{{0.001459, 5}, {0.006338, 20}, {0.016207, 51}, {0.037493, 105}, {0.076312, 190},
 {0.128472, 315}, {0.216543, 490}, {0.386452, 726}, {0.697105, 1035}, {0.947616, 1430}}
```

In[235]:=
```
Flist[5] = experiment[#, 5] & /@ Range[10]
```

26.4657 Second

Out[235]=
```
{{0.002552, 7}, {0.016583, 36}, {0.059041, 108},
 {0.168279, 252}, {0.395561, 506}, {0.857346, 918}, {1.602924, 1547},
 {2.933435, 2464}, {5.042421, 3753}, {8.547611, 5512}}
```

In[236]:=
```
Flist[6] = experiment[#, 6] & /@ Range[10]
```

174.911 Second

Out[236]=
```
{{0.004004, 11}, {0.040087, 65}, {0.195557, 221},
 {0.701540, 574}, {2.025705, 1265}, {4.913787, 2492}, {10.945729, 4522},
 {23.051759, 7704}, {44.932045, 12483}, {83.759705, 19415}}
```

In[237]:=
```
Flist[7] = experiment[#, 7] & /@ Range[8]
```

330.277 Second

Out[237]=
```
{{0.005622, 15}, {0.089162, 110}, {0.681502, 429}, {3.055511, 1240},
 {10.884433, 2990}, {33.225293, 6372}, {82.240041, 12405}, {196.821582, 22528}}
```

In[238]:=
```
Flist[8] = experiment[#, 8] & /@ Range[6]
```

302.719 Second

Out[238]=
```
{{0.010850, 22}, {0.267966, 185}, {2.461819, 810},
 {14.321746, 2580}, {61.641596, 6765}, {222.491062, 15525}}
```

In[239]:=
```
Flist[9] = experiment[#, 9] & /@ Range[5]
```

415.602 Second

Out[239]=
```
{{0.013778, 30}, {0.672458, 300},
 {8.219750, 1479}, {61.320062, 5180}, {348.810322, 14725}}
```

In[240]:=
```
Flist[10] = experiment[#, 10] & /@ Range[4]
```

381.524 Second

Out[240]=
```
{{0.215057, 42}, {2.882951, 481}, {34.553778, 2640}, {349.965110, 10108}}
```

In[241]:=
```
MaxMemoryUsed[]
```

Out[241]=
```
1048894992
```

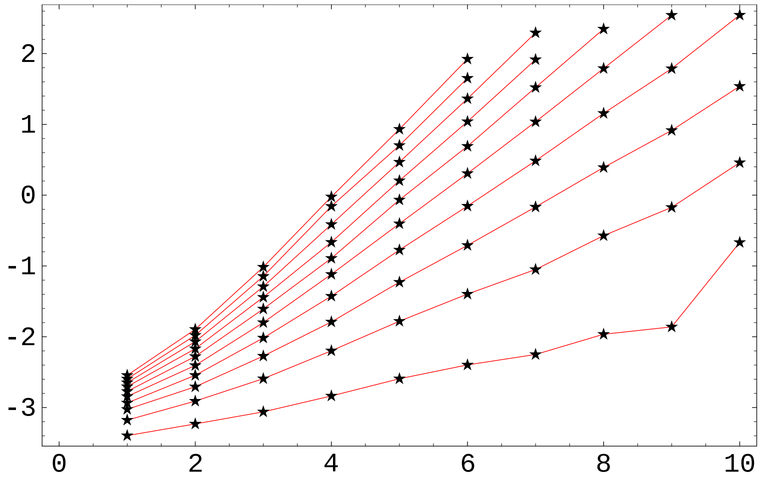
Graphs:

In[242]:=

```
tlist = Transpose[PadRight[#, 10, Null] & /@ Array[Flist, {10}]] /. Null → Sequence[];
```

In[243]:=

```
Ftimingsplot = MultipleListPlot[Log[10, Map[First, tlist, {2}]], PlotJoined → True,
  Axes → False, Frame → True, SymbolShape → PlotSymbol[Star], PlotStyle → Hue[0]]
```

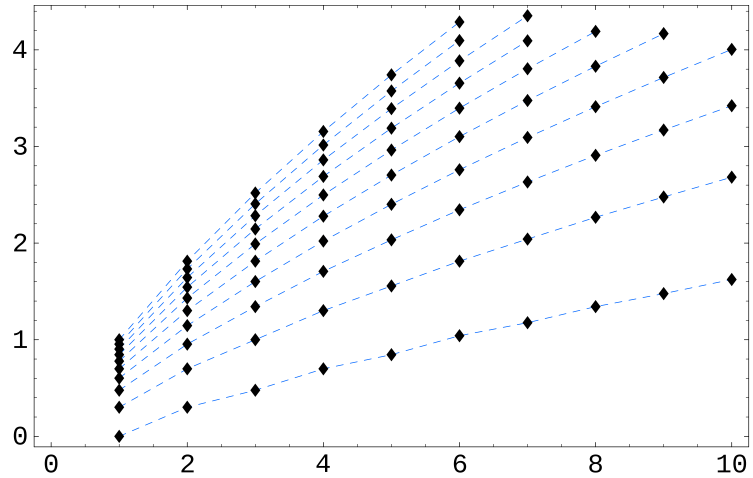


Out[243]=

```
- Graphics -
```

In[244]:=

```
Ftermsplot = MultipleListPlot[Log[10, Map[Last, tlist, {2}]],
  PlotJoined → True, Axes → False, Frame → True, SymbolShape → PlotSymbol[Diamond],
  PlotStyle → {{Hue[0.6], Dashing[{0.01, 0.01}]}}]
```

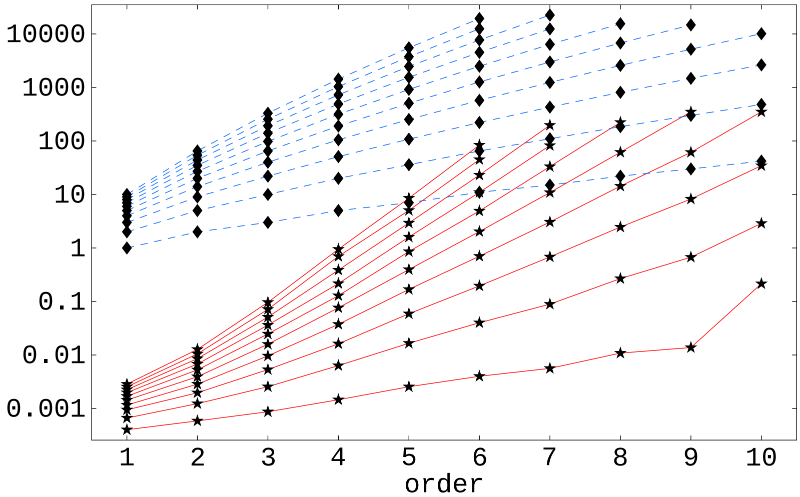


Out[244]=

```
- Graphics -
```

In[245]:=

```
Fplot = Show[Ftimingsplot, Ftermsplot, FrameLabel → {order, None},
  FrameTicks → ticks, PlotRange → {{0.5, 10.5}, All}, ImageSize → 300]
```



Out[245]=

```
- Graphics -
```

■ 8. Final comments

Note: For further information about `xPert‘`, and to be kept informed about new releases, you may contact the authors electronically at brizuela@iem.cfmac.csic.es, jmm@iem.cfmac.csic.es and mena@iem.cfmac.csic.es. Sugges−tions and comments are welcome and very much appreciated!

This is xPertDoc.nb, the docfile of `xPert‘`, currently in version 1.0.0.

In[246]:=

```
?xAct‘xPert‘*
```

xAct‘xPert‘

DefMetricPerturbation	GaugeChange	Perturbed
DefTensorPerturbation	GeneralPerturbation	$PerturbationParameter
Disclaimer	Perturbation	$Version
ExpandPerturbation	PerturbationOrder	$xTensorVersionExpected

```
0.180011 Second
```